HEAT AND GEOMETRY

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1. 2018.07.09 Post-Lecture Notes

We shall start in the general context of $\mathbb{R}^n$. We define the Laplace operator in $\mathbb{R}^n$ to be

$$-\sum_{i=1}^n \partial_{x_i}^2.$$

The *heat operator*

$$\Xi := \partial_t + \Delta.$$

On a Riemannian manifold (M, g) we will use

$$\Delta = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^n \partial_{x_i} \sqrt{\det g} g^{ij} \partial_{x_j}, \quad \Xi := \partial_t + \Delta.$$

This coincides with the Euclidean case, because there the Riemannian metric g is in terms of the standard local coordinates equal to the identity matrix at every point of $\mathbb{R}^n$, so we obtain the same expression as above. Of course, you can screw around with other coordinates (polar coordinates can be useful at times), and the operator will look different, but it is indeed the same operator. You can check this as an **exercise** if you like. We use the symbol Ξ for the heat operator because it is the closest natural LaTex symbol to the Chinese symbol for sun. The sun is pretty hot. If you want to use your own symbol, that's fine, you just need to define it precisely and be consistent throughout your text.

1.1. The initial value problem (IVP) for the heat equation on $\mathbb{R}^n$.

Let's begin by solving the initial value problem for the heat equation in $\mathbb{R}^n$. Let $f \in \mathcal{L}^2(\mathbb{R}^n) \cap \mathcal{C}_b(\mathbb{R}^n)$ be a square integrable, continuous, and bounded function. We wish to solve:

$$\Xi u(x,t) = 0, \quad t > 0, \quad x \in \mathbb{R}^n,$$

with the initial condition

$$u(x, 0) = f(x).$$

Physically this means that $f(x)$ is the distribution of heat in our $\mathbb{R}^n$ universe at the start of time. We're not considering the big bang, so it is natural to assume that f is bounded and continuous. Square integrability is also convenient. Perhaps you've solved this before? They key tool is the Fourier transform which maps $\mathcal{L}^2(\mathbb{R}^n)$ to itself.

Definition 1. The Fourier transform on $\mathbb{R}^n$ is defined for $\varphi \in \mathcal{L}^1(\mathbb{R}^n)$ by

$$\mathcal{F}(\varphi)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} \varphi(x) dx = \hat{\varphi}(\xi).$$

Exercise 1. *Check that for any $\varphi \in \mathcal{L}^1(\mathbb{R}^n)$ its Fourier transform is well-defined. Then, look up the theorem which states that there is a unique, well-defined extension to $\mathcal{L}^2$. If you're really ambitious, sketch the proof.*

Exercise 2. *Give an example of a function in $\mathcal{L}^1(\mathbb{R}^n)$ but not in $\mathcal{L}^2(\mathbb{R}^n)$ and vice versa. Give an example of a function which is in both. What happens if you consider instead $\mathcal{L}^1(X)$ and $\mathcal{L}^2(X)$ when X is a compact subset of $\mathbb{R}^n$?*

Proposition 2. *Assume that the partial derivative $\partial_j \varphi$ is Fourier-transformable (it is sufficient to lie in $\mathcal{L}^1$ and/or $\mathcal{L}^2$). Then*

$$\mathcal{F}(\partial_j \varphi)(\xi) = 2\pi i \xi_j \hat{\varphi}(\xi).$$

Proof: WLOG assume that $j = 1$. Then

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{-2\pi i \sum_1^n x_j \xi_j} \partial_1 \varphi(x) dx_1 \dots dx_n$$

$$= \lim_{x_1 \to \infty} \int_{\mathbb{R}^{n-1}} e^{-2\pi i \sum_1^n x_j \xi_j} \varphi(x) dx - \lim_{x_1 \to -\infty} \int_{\mathbb{R}^{n-1}} e^{-2\pi i \sum_1^n x_j \xi_j} \varphi(x) dx$$

$$-(-2\pi i \xi_1) \int_{\mathbb{R}^n} e^{-2\pi i \sum_1^n x_j \xi_j} \varphi(x) dx.$$

Exercise: Kill off the boundary terms to complete the proof.

Corollary 3. *Assume that $\partial_j \varphi$ and $\Delta \varphi$ are Fourier transformable. Then*

$$\mathcal{F}(\Delta \varphi)(\xi) = 4\pi 2 |\xi|^2 \hat{\varphi}(\xi).$$

Proof: Apply the proposition twice to obtain:

$$\mathcal{F}(-\sum_1^n \partial_j^2 \varphi)(\xi) = \sum_1^n -\hat{\varphi}(\xi)(2\pi i \xi_j)^2 = 4\pi^2 |\xi|^2 \hat{\varphi}(\xi).$$

Definition 4. For two elements of $\mathcal{L}^2(\mathbb{R}^n)$ define the convolution by

$$f * g(x) := \int_{\mathbb{R}^n} f(x-y) g(y) dy.$$

Proposition 5. *As long as f and g are in $\mathcal{L}^2(\mathbb{R}^n)$ the convolution is well-defined. Convolution is commutative, that is*

$$f * g(x) = g * f(x).$$

Moreover,

$$\mathcal{F}(f * g)(\xi) = \hat{f}(\xi) \hat{g}(\xi).$$

Proof: We first estimate

$$|f * g(x)| = \left| \int_{\mathbb{R}^n} f(x-y) g(y) dy \right| \le \int_{\mathbb{R}^n} |f(x-y)||g(y)| dy.$$

The point $x \in \mathbb{R}^n$ is fixed and arbitrary, so I define a function

$$\phi(y) = f(x-y).$$

Then

$$|f * g(x)| \leq \int_{\mathbb{R}^n} |\phi(y)||g(y)|dy \leq ||\phi||||g||.$$

We compute

$$||\phi||^2 = \int_{\mathbb{R}^n} |f(x-y)|^2 dy = \int_{\mathbb{R}^n} |f(t)|^2 dt = ||f||^2.$$

So, in the end we see that

$$|f * g(x)| \leq ||f||||g||,$$

where these are the $\mathcal{L}^2$ norms. Next,

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

We want to get $g(x-z)$ so we define

$$y = x - z \implies x - y = z.$$

The Jacobian factor when we change the integral is just one, so we get

$$f * g(x) = \int f(z)g(x-z)dz = \int g(x-z)f(z)dz = g * f(x).$$

Finally,

$$\mathcal{F}(f * g)(\xi) = \int f * g(x)e^{-2\pi i x\cdot\xi}dx = \int\int f(x-y)g(y)e^{-2\pi i x\cdot\xi}dydx.$$

We do a little sneaky trick

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)e^{-2\pi i x\cdot\xi}e^{-2\pi i y\cdot\xi}e^{2\pi i y\cdot\xi}dydx$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)e^{-2\pi i(x-y)\cdot\xi}g(y)e^{-2\pi i y\xi}dydx.$$

Let $z = x - y$. Then $dz = -dy$ so

$$= \int\int f(z)e^{-i2\pi z\cdot\xi}dzg(y)e^{-2\pi i y\cdot\xi}dy = \int\int f(z)e^{-i2\pi z\cdot\xi}dzg(y)e^{-2\pi i y\cdot\xi}dy$$

$$= \hat{f}(\xi)\hat{g}(\xi).$$

Now we can solve the IVP for the heat equation. We shall assume that everything is well-defined and transformable as necessary, and in the end verify that this is indeed the case. So, we hit the heat equation with the Fourier transform:

$$\mathcal{F}(\Xi u) = 0 \implies \partial_t\hat{u}(\xi,t) + 4\pi^2|\xi|^2\hat{u}(\xi,t).$$

Moreover the initial condition is Fourier transformable, so we must have

$$\hat{u}(\xi,0) = \hat{f}(\xi).$$

The Fourier transformed heat equation has metamorphed into an elementary ODE in the t variable. Its unique solution is

$$\hat{u}(\xi,t) = \hat{f}(\xi)e^{-4\pi^2|\xi|^2 t}.$$

So, we have found the Fourier transform of our solution. Since we know how Fourier transform interacts with convolutions, we shall have found our solution to the original IVP if we can find a function whose Fourier transform is $e^{-4\pi^2|\xi|^2 t}$.

Proposition 6. *For a function*

$$\phi(x) = e^{-a|x|^2}, \quad a > 0$$

its Fourier transform is

$$\left(\frac{\pi}{a}\right)^{n/2} e^{-\pi^2|\xi|^2/a}.$$

Proof: By definition

$$\hat{\phi}(\xi) = \int e^{-2\pi i x\cdot\xi} e^{-a|x|^2} dx = \int e^{-a|x|^2 - 2\pi i x\cdot\xi} dx.$$

We write the exponent as

$$-\left(\sum_{j=1}^n (\sqrt{a}x_j)^2 + 2\sqrt{a}x_j i\xi_j \frac{\pi}{\sqrt{a}} + \left(\frac{\pi i \xi_j}{\sqrt{a}}\right)^2 - \left(\frac{\pi i \xi_j}{\sqrt{a}}\right)^2\right).$$

We note that

$$e^{\sum_1^n \cdots} = \prod_1^n e^{\cdots}.$$

Moreover everything converges beautifully, so it suffices to compute each of these integrals in one dimension. Thus we compute

$$\star = \int_{\mathbb{R}} e^{-(\sqrt{a}y + \pi i \xi/\sqrt{a})^2} dy.$$

Here we use a bit of complex analysis. Let $z = \sqrt{a}y + i\pi\xi/\sqrt{a} = z(y)$ then

$$\dot{z}(y) = \sqrt{a}dy, \implies \star = \frac{1}{\sqrt{a}} \int_\Gamma e^{-z(y)^2} dz,$$

where Γ is a contour in the complex plane. The specific contour we choose goes along the real axis horizontally between two very large values, like $\pm R$. It proceeds to make a box shape either going above or below the real axis depending on the sign of ξ. If ξ is positive then we go above, if $\xi < 0$ then we go below. Since our function is entire, the integral over the box vanishes. Moreover, the integrals along the sides of the box tend to zero as $R \to \infty$. So, letting $R \to \infty$, we get that the integral with $\Im z = i\pi\xi/\sqrt{a}$ is the same as the integral with $\Im z = 0$. In this case, the integral is just

$$\frac{1}{\sqrt{a}} \int e^{-ay^2} dz = \int e^{-ay^2} dy = \sqrt{\frac{\pi}{a}}.$$

Therefore, the term

$$\int e^{-\left((\sqrt{a}x_j)^2 + 2\sqrt{a}x_j i\xi_j \frac{\pi}{\sqrt{a}} + \left(\frac{\pi i \xi_j}{\sqrt{a}}\right)^2 - \left(\frac{\pi i \xi_j}{\sqrt{a}}\right)^2\right)} dx_j$$

is

$$e^{(\pi i \xi_j)^2/a} \int e^{-(\sqrt{a}x_j + i\xi_j \frac{\pi}{\sqrt{a}})^2} dy = e^{-\frac{\pi^2 \xi_j^2}{a}} \sqrt{\frac{\pi}{a}}.$$

The full integral is the product over $j = 1, \ldots, n$ of these, which gives

$$\prod_{j=1}^n e^{-\frac{\pi^2\xi_j^2}{a}} \sqrt{\frac{\pi}{a}} = \left(\frac{\pi}{a}\right)^{n/2} e^{-\pi^2|\xi|^2/a}.$$

We therefore see that to obtain $-4\pi^2|\xi|^2 t$ in the exponent, we need to have

$$4t = \frac{1}{a} \iff a = \frac{1}{4t}.$$

However, we don't have a factor of

$$\left(\frac{\pi}{a}\right)^{n/2}$$

in front. That's okay because in the eyes of the Fourier transform, this stuff is constant. So,

$$\mathcal{F}\left(\left(\frac{a}{\pi}\right)^{n/2} e^{-a|x|^2}\right) = e^{-\pi^2|\xi|^2/a}.$$

Therefore, the Fourier transform of

$$\frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)} \text{ is } e^{-4\pi^2|\xi|^2 t}.$$

Consequently, the solution to the IVP for the heat equation is given by the convolution

$$u(x,t) = \int_{\mathbb{R}^n} f(x-y)\frac{1}{(4\pi t)^{n/2}} e^{-|y|^2/(4t)} dy = \int_{\mathbb{R}^n} f(y)\frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/(4t)} dy.$$

For all $t > 0$ this is well-defined. Moreover, one can prove that since we assumed that f is continuous and bounded

ic (1.1)
$$\lim_{t\downarrow 0} \int_{\mathbb{R}^n} f(y)\frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/(4t)} dy = f(x) \quad \forall x \in \mathbb{R}.$$

I leave this as an **exercise.** Actually, it is a specific example of a more general fact.

Exercise 3. *Let g be an $\mathcal{L}^1$ function which satisfies*

$$\int_{\mathbb{R}^n} g(x)dx = 1.$$

Let f be a bounded, continuous function. Then for any $x \in \mathbb{R}^n$, prove that:

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} f(x-y)g(y/\varepsilon)\varepsilon^{-n} dy = f(x).$$

Use this to prove (ic 1.1) above. For the case of $\mathbb{R}^1$, see §convolutionapprox ?? below.

We have therefore found the heat kernel on $\mathbb{R}^n$.

Definition 7. The heat kernel on $\mathbb{R}^n$ is the Schwartz kernel of the fundamental solution to the heat equation and is given by

$$H(x,y,t) = e^{-\frac{|x-y|^2}{4t}}(4\pi t)^{-n/2}, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n, \quad t \in (0,\infty).$$

In general, the heat kernel for the Laplacian on a Riemannian manifold, M, or a domain in $\mathbb{R}^n$ acting on a particular Hilbert space, H, is a function on

$$M \times M \times (0,\infty)$$

which satisfies (here the operator, Δ can act in either the x or y variable)

$$\Xi H(x,y,t) = 0, \quad t > 0,$$

and

$$\lim_{t\downarrow 0} \int_M H(x,y,t)f(y)dy = f(x) \quad \forall f \in H.$$

We shall see that in general, the heat kernel on a domain, M, in $\mathbb{R}^n$ or a Riemannian manifold (M, g) is also the Schwartz kernel of the fundamental solution to the heat equation. It is a smooth function on $M \times M \times (0, \infty)$ which satisfies

$$H(x, y, t) = H(y, x, t),$$
$$\Xi H(x, y, t) = 0 \quad \forall t > 0,$$
$$H(x, y, 0) = \delta(x - y), \text{ as distributions.}$$

Since we cannot explicitly compute these heat kernels in the same way as on $\mathbb{R}^n$, it is helpful to keep our $\mathbb{R}^n$ heat kernel in mind as an example. We shall see that many of the same fundamental properties are shared by those heat kernels which we cannot explicitly compute, thus even though the $\mathbb{R}^n$ case may seem “trivial” (I hate that word), it's really quite instructive!

1.2. **Properties of the Euclidean heat kernel.**

(1) For $x \neq y$ the heat kernel decays *rapidly* as $t \downarrow 0$, that is
$$\lim_{t \downarrow 0} \frac{H(x, y, t)}{t^N} = 0 \quad \forall N \in \mathbb{N}.$$
This is known as *rapid off-diagonal decay.*
(2) When $x = y$, however,
$$H(x, x, t) = O(t^{-n/2}), \quad t \downarrow 0.$$
(3) $H(x, y, t) > 0$ for all x and y in $\mathbb{R}^n$.
(4) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then,
$$\int_\Omega H(x, x, t) dx = \frac{|\Omega|}{(4\pi t)^{n/2}}.$$

1.3. **Heat spaces.** Heat kernels are in general defined on $M \times M \times [0, \infty)$. Let's first consider the case $M = \mathbb{R}^n$. Let's draw a picture. The heat kernel is super lovely except in one particular spot... The set

$$\{(x, x, 0) : x \in \mathbb{R}^n\}.$$

This is an n-dimensional subset of $M \times M \times [0, \infty)$. At this set the heat kernel is discontinuous. Everywhere else it's smooth, but here it is a distribution (which is demonstrably not given by the action of a function, see the exercise below).

Exercise 4. *Prove that there is no $u \in \mathcal{L}^2(\mathbb{R}^n)$ which satisfies*

$$\int_{\mathbb{R}^n} u(x - y) f(y) dy = f(x),$$

for all f which are smooth and compactly supported. In this way, the δ distribution cannot be represented by any $\mathcal{L}^2$ function.

To be able to talk about heat spaces, we need the notion of a manifold with corners.

Definition 8. An n-dimensional manifold with corners, Z, is a topological space locally modelled on neighborhoods $[0, \infty)^k \times \mathbb{R}^{n-k}$ for various $k \in \{0, 1, \dots, n\}$, in the same sense that a manifold is modelled on neighborhoods of $\mathbb{R}^n$. A boundary defining function for a boundary hypersurface is a function $\rho : Z \to [0, \infty)$ such that $\rho^{-1}(0)$ is equal to precisely that hypersurface, and $d\rho \neq 0$ at that hypersurface.

We shall consider blowups along *p-submanifolds.* An embedded submanifold Y contained in a manifold with corners, Z, is a *p-submanifold* if near each point $q \in Y$, there exist local coordinates so that Y is defined by the vanishing of a subset of these local coordinates. For example, the boundary faces of Z are p-submanifolds. The intersection of two or more boundary faces of Z is also a p-submanifold. The normal blowup of Z around Y is denoted by

$$[Z; Y] = \mathrm{ff} \sqcup (Z \setminus Y).$$

Above, ff is the inward pointing spherical normal bundle of Y which has replaced Y in $[Z; Y]$. There is a unique minimal differentiable structure with respect to which $[Z; Y]$ is a manifold with corners such that the following hold.

(1) There is a "blow-down" map,

$$\beta : [Z; Y] \to Z$$

such that smooth functions on Z lift under β to smooth functions on $[Z; Y]$, that is if $f \in \mathcal{C}^\infty(Z)$ then $\beta^*(f) = f \circ \beta$ is a smooth function on $[Z; Y]$.

(2) Polar coordinates around Y are smooth functions in $[Z; Y]$.

In case we wish to blow up two or more p-submanifolds, we write

$$[Z; Y_1; Y_2]$$

to indicate that we first blow up Y_1 and next blow up Y_2.

When studying the heat equation and the heat kernel, we will want to use *parabolic* blowups. For our purposes, all parabolic blowups will be parabolic in the t direction and occur at p-submanifolds contained in $\{t = 0\}$. Such a t-parabolic blowup around a p-submanifold, Y, shall be denoted by

$$[Z; Y, dt] = tf \sqcup (Z \setminus Y).$$

Above, the submanifold, Y is replaced by tf, which is the t-parabolic inward pointing normal bundle of Y. If the set Y is defined by the vanishing of t and x, that is $Y = \{t = 0\} \cap \{x = 0\}$, so that local coordinates near Y are given by (t, x, y), then the smooth structure near tf is generated by functions which are homogeneous of non-negative integer order with respect to the map

$$(t, x, y) \mapsto (a^2 t, ax, y), \quad a > 0.$$

For further details concerning parabolic blowups, we refer to [2, p. 252–259] and [1, §2.1–2.2].

To create the heat space for $M = \mathbb{R}^n$, we *blow up* the set at which the heat kernel is discontinuous. The heat space is, with the notation above,

$$X_h^2 = [\mathbb{R}^n \times \mathbb{R}^n \times [0, \infty); \mathrm{diag} \times \{t = 0\}, dt], \quad \mathrm{diag} = \{(x, x) \in \mathbb{R}^n \times \mathbb{R}^n\}.$$

In this example, there is a set of local coordinates which are particularly helpful for understanding the heat space:

$$x, x' \in \mathbb{R}^n, \quad rw' := x - x', \quad t = r^2 w_0, \quad w = (w_0, w') \in S,$$

where

$$S = \{w \in \mathbb{R}^{n+1} : w_0 \geq 0, \quad w_0^2 + |w'|^4 = 1\}.$$

The *heat space,* X_h^2 for $X = \mathbb{R}^n$ is then equivalently given by

$$[0, \infty) \times S \times \mathbb{R}^n.$$

It is a manifold with two boundary hypersurfaces sitting at $t = 0$ denoted by tf, which is the face obtained by blowing up, and tb which is the other face. The blow down map in terms of the coordinates above sends

$$X_h^2 \to \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty), \quad (w, y, r) \mapsto (rw' + y, y, r^2 w_0).$$

The face tf has r as a boundary defining function. The boundary face

$$tb = \beta^{-1}(\{t = 0\} \setminus \text{diag} \times \{t = 0\}).$$

For a simple example, let's take $n = 1$. Then our local coordinates are

$$x - x' = rw', t = r^2 w_0,$$

and

$$S = \{(w_0, w') : w_0 \geq 0, w_0^2 + (w')^4 = 1\}.$$

The heat space is

$$[0, \infty) \times S \times \mathbb{R}.$$

The functions w_0 and r define tb and tf, respectively. Then, the lift of t to X_h^2 is $w_0 r^2$.

Exercise 5. *Show that the definition of X_h^2 in terms of the parabolic blowup is equal to the more concrete formulation as $[0, \infty) \times S \times \mathbb{R}^n$.*

References

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