HEAT AND GEOMETRY

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We begin by showing that the heat kernel mollifies functions.

Proposition 1. *Let M be a compact Riemannian manifold with heat kernel $H(t,x,y)$. Assume that $f \in \mathcal{L}^2(M)$. Then*

$$u(x,t) := \int_M H(t,x,y) f(y) dy$$

is smooth in both t and x for all $t > 0$.

Proof: For all fixed $t > 0$, by the smoothness of the heat kernel, $H \in \mathcal{L}^\infty(M)$, and so $|H(t,x,y)f(y)| \leq N|f(y)|$ for some constant N. Moreover, since M is compact, $\mathcal{L}^2(M) \subset \mathcal{L}^1(M)$ and in general $\mathcal{L}^p(M) \subset \mathcal{L}^q(M)$ whenever $p > q \geq 1$. Similarly, the regularity of H shows that we can choose N so that we also have

$$|\partial_x H(t,x,y) f(y)| \leq N|f(y)| \quad \forall y \in M.$$

We may therefore use the dominated convergence theorem to say that

$$\partial_x u(x,t) = \int_M \partial_x H(t,x,y) f(y) dy,$$

and this is well defined for all $x \in M$. A similar DCT argument shows that u is also differentiable in t. We can repeat this argument as many times as we like, due to the smoothness of H in both x and t, to obtain as many derivatives of u in x and t as we like. Hence smoothness.

Corollary 2. *Let us denote by*

$$e^{-t\Delta}$$

the operator which acts on $\mathcal{L}^2(M)$ by

$$e^{-t\Delta} f \mapsto \int_M H(t,x,y) f(y) dy,$$

where H is the heat kernel on M, our compact Riemannian manifold. Then $e^{-t\Delta}$ is a compact operator for each $t > 0$.

Proof: We have for each $t > 0$ that $e^{-t\Delta} : \mathcal{L}^2(M) \to \mathcal{C}^\infty((0,\infty) \times M)$. For each fixed $t > 0$, we therefore get

$$e^{-t\Delta} : \mathcal{L}^2(M) \to \mathcal{C}^\infty(M) \subset H^1(M) \hookrightarrow \mathcal{L}^2(M),$$

where the last map is the identity map. By the Sobolev embedding theorem (or is it Rellich something or other? I am always confusing these), the identity map from H^1 to $\mathcal{L}^2$ is compact.

Proposition 3. *The operator $e^{-t\Delta}$ is self-adjoint on $\mathcal{L}^2(M)$.*

Proof: Let u and v be in $\mathcal{L}^2(M)$. For now we shall drop the dependence on $t > 0$. Then we compute

$$\langle e^{-t\Delta} u, v \rangle = \int_M e^{-t\Delta} u(x) \overline{v(x)} dx = \int_M \int_M H(t,x,y) u(y) \overline{v(x)} dy dx$$
$$= \int_M \int_M \overline{H(t,x,y) v(x)} dx u(y) dy$$
$$= \int_M u(y) \overline{e^{-t\Delta} v(y)} dy = \langle u, e^{-t\Delta} v \rangle.$$

This holds for all u and v in $\mathcal{L}^2(M)$, so indeed, $e^{-t\Delta}$ is its own adjoint.

Proposition 4. *The operator $e^{-t\Delta}$ satisfies the semi-group property, that is*

$$e^{-\tau\Delta} e^{-s\Delta} = e^{-(\tau+s)\Delta}, \quad \forall \tau, s > 0.$$

Proof: Let us fix $t > s > 0$, and let $f \in \mathcal{L}^2(M)$, then

$$e^{-(t-s)\Delta} e^{-s\Delta} f(x) = e^{-(t-s)\Delta} \int (M H(s,z,y) f(y) dy)(x)$$

$$= \int_M H(t-s,x,z) (\int_M H(s,z,y) f(y) dy) dz = \int_M \int_M H(t-s,x,z) H(s,z,y) dz) f(y) dy.$$

Consequently, the action of the operator $e^{-(t-s)\Delta} e^{-s\Delta}$ is given by the integral kernel

$$\int_M H(t-s,x,z) H(s,z,y) dz.$$

For $t > 0$ we can apply

$$(\partial_t + \Delta_x) \int_M H(t-s,x,z) H(s,z,y) dz = 0.$$

Moreover,

$$\lim_{t \to 0} \int_M (\int_M H(t-s,x,z) H(s,z,y) dz) f(y) dy = \lim_{t \to 0} \int_M H(t-s,x,z) (\lim_{s \to 0} \int_M H(s,z,y) f(y) dy) dz.$$

This follows because $t > s > 0$ so that when $t \to 0$ this forces $s \to 0$ as well. So, computing the s limit, by definition of the heat kernel, the inside limit is $f(z)$, and so we get

$$\lim_{t \to 0} \int_M H(t,x,z) f(z) dz = f(x).$$

By the regularity of H, the function,

$$\tilde{H}(t,x,y) = \int_M H(t-s,x,z) H(s,z,y) dz$$

satisfies the definition of being a heat kernel. By the uniqueness of the heat kernel, we therefore have

$$\tilde{H}(t,x,y) = H(t,x,y).$$

Consequently, the operator $e^{-(t-s)\Delta}e^{-s\Delta}$ acts by

$$e^{-(t-s)\Delta}e^{-s\Delta}f(x) = \int_M H(t,x,y)f(y)dy = e^{-t\Delta}f(x).$$

So, we have proven that the operators

$$e^{-(t-s)\Delta}e^{-s\Delta} = e^{-t\Delta}.$$

In particular, for letting $\tau = t - s > 0$, so that $t = \tau + s$, we have proven

$$e^{-\tau\Delta}e^{-s\Delta} = e^{-(\tau+s)\Delta}.$$

We can now apply the spectral theorem for compact self-adjoint operators to the operator induced by the heat kernel, $e^{-t\Delta}$.

Theorem 5. *For each $t > 0$ there is an orthonormal basis of $\mathcal{L}^2(M)$ consisting of eigenfunctions of $e^{-t\Delta}$ with eigenvalues $\gamma_i(t)$ such that $\gamma_i(t) \to 0$ as $i \to \infty$. Moreover, $\gamma_i(t) > 0$ for all i, and in fact, the eigenfunctions are independent of t. Each $\gamma_i(t) = e^{-\lambda_i t}$, where the eigenfunction w_i is an eigenfunction for Δ with eigenvalue λ_i. The eigenvalues*

$$0 = \lambda_0 < \lambda_1 \leq \ldots \uparrow \infty$$

accumulate only at ∞.

Proof: The first part of the theorem is simply the spectral theorem for self-adjoint compact operators acting on Hilbert spaces. Let us prove that the eigenvalues $\gamma_i(t)$ are positive. To do this we use the semi-group property and self-adjointness:

$$\langle e^{-t\Delta}f, f\rangle = \langle e^{-\frac{1}{2}\Delta}e^{-\frac{1}{2}\Delta}f, f\rangle = \langle e^{-\frac{1}{2}\Delta}f, e^{-\frac{1}{2}\Delta}f\rangle \geq 0.$$

So, if f is an eigenfunction, then

$$e^{-t\Delta}f = \gamma_i(t)f \implies \langle e^{-t\Delta}f, f\rangle = \gamma_i(t)||f||^2 \geq 0 \implies \gamma_i(t) \geq 0.$$

We wish to show that 0 is not an eigenvalue. Let us do so by contradiction. If it were, then there would be a non-zero $f \in \mathcal{L}^2(M)$ such that

$$e^{-t\Delta}f = 0 \implies 0 = ||e^{-t/2\Delta}f|| = ||e^{-t/4\Delta}f|| = \ldots = ||e^{-2^{-N}\Delta}f||, \quad \forall N \in \mathbb{N}.$$

Since

$$\lim_{t\downarrow 0} e^{-t\Delta}f(x) = f(x) \implies \lim_{t\downarrow 0} ||e^{-t\Delta}f|| = ||f||,$$

which shows that

$$||f|| = 0 \implies f = 0.$$

This is a contradiction. Therefore, all of the eigenvalues $\gamma_i(t)$ for $e^{-t\Delta}$ are positive. Let $w_i(t)$ be the corresponding eigenfunction. Then,

$$e^{-t\Delta}w_i(t) = \gamma_i(t)w_i(t).$$

Moreover, by the semi-group property

$$e^{-s\Delta}e^{-t\Delta} = e^{-(s+t)\Delta} = e^{-(t+s)\Delta} = e^{-t\Delta}e^{-s\Delta}.$$

Consequently,

$$e^{-s\Delta}e^{-t\Delta}w_i(t) = e^{-(t+s)\Delta}w_i(t) = e^{-s\Delta}\gamma_i(t)w_i(t) = \gamma_i(t)e^{-s\Delta}w_i(t).$$

On the other hand

$$e^{-s\Delta}e^{-t\Delta} = e^{-t\Delta}e^{-s\Delta} \implies e^{-t\Delta}e^{-s\Delta}w_i(t) = \gamma_i(t)e^{-s\Delta}w_i(t).$$

This shows that for all $s > 0$, $e^{-s\Delta}w_i(t)$ is an eigenfunction for $e^{-t\Delta}$ with the same eigenvalue, $\gamma_i(t)$ as $w_i(t)$ has. Therefore, $e^{-s\Delta}w_i(t)$ is given by a linear combination of the eigenfunctions at time t for the eigenvalue $\gamma_i(t)$. Since t is arbitrary, there is a basis of eigenfunctions for $\mathcal{L}^2(M)$ which is independent of t. Let us continue writing these by w_i. Then, by the semi-group property

$$e^{-s\Delta}e^{-t\Delta}w_i = \gamma_i(t)e^{-s\Delta}w_i = \gamma_i(t)\gamma_i(s) = \gamma_i(t+s).$$

Moreover, by definition, we have

$$0 = (\partial_t + \Delta)e^{-t\Delta}w_i = (\partial_t + \Delta)(\gamma_i(t)w_i) = \dot{\gamma}_i(t)w_i + \gamma_i(t)\Delta w_i.$$

Therefore

$$\Delta w_i = -\frac{\dot{\gamma}_i(t)}{\gamma_i(t)}w_i.$$

We use an oldie but a goodie: the left side is independent of t. Hence the right side must be also. Hence,

$$-\frac{\dot{\gamma}_i(t)}{\gamma_i(t)} = \lambda_i \in \mathbb{R}.$$

This is an ODE for the eigenvalue $\gamma_i(t)$, whose solution is given by

$$\gamma_i(t) = C'e^{-\lambda_i t},$$

for some constant C'. To determine the constant, we think about the initial condition. As $t \to 0$, we have $e^{-t\Delta}w_i \to w_i$. Therefore, $\gamma_i(t) \to 1$ as $t \to 0$. Consequently $C' = 1$. We have therefore shown that there are real numbers λ_i such that

$$\gamma_i(t) = \lambda_i.$$

Furthermore, the functions w_i satisfy

$$\Delta w_i = \lambda_i w_i.$$

Since $w_i \in \mathcal{L}^2$, this shows that $\Delta w_i \in \mathcal{L}^2$ so actually $w_i \in H^2$. Repeating, we get that w_i are smooth (this is called boot-strapping in some circles). We can also use the fact that M is closed to compute using Stokes's theorem (integration by parts) that

$$\langle \Delta w_i, w_i \rangle = \lambda_i ||w_i||^2 = \int_M |\nabla w_i|^2 \geq 0.$$

This shows that $\lambda_i \geq 0$.

Exercise 1. *Prove that $e^{-t\Delta}$ is injective on $\mathcal{L}^2(M)$ for each $t > 0$.*

The value 0 is an eigenvalue for Δ with eigenfunction equal to a constant function.

Exercise 2. *Prove that the constant function is not an eigenfunction for $e^{-t\Delta}$ for any $t > 0$.*

The eigenfunctions $\{w_i\}$ for $e^{-t\Delta}$ span $\mathcal{L}^2(M)$, yet none of these are the constant function. It's okay, there's no contradiction, because the constant function is just some linear combination of these. Indeed, as long as

$$\int_M w_i \neq 0,$$

for some i, this shows that the constant function is not orthogonal to the set $\{w_i\}$, so there is no contradiction to these forming an ONB for $\mathcal{L}^2$. However, the standard way in which we choose our ONB for $\mathcal{L}^2$ is to choose the ONB for the Laplace operator beginning with the constant function. The convention with the λs and the ws is to order them in the following way: start with $\lambda_0 = 0$. Then normalize the constant function ϕ_0 to have $\mathcal{L}^2$ norm equal to one. Next, continue with the λ's which, since $\gamma_i(t) = e^{-\lambda_i t}$, and these tend to 0 as $i \to \infty$, it follows that the λ_i accumulate only at ∞. So, for $\lambda_1 > 0 = \lambda_0$, choose an eigenfunction ϕ_1 which has $\mathcal{L}^2$ norm equal to one and which is orthogonal to ϕ_0. Continue.

At this point we can formally define the function

$$e(t, x, y) = \sum_{k \geq 0} e^{-\lambda_k t} \phi_k(x) \phi_k(y).$$

Exercise 3. *Let $f \in \mathcal{L}^2(M)$. Show that*

$$\int_M e(t, x, y) f(y) dy \in \mathcal{L}^2(M).$$

Show that in fact $e(t, x, y) = H(t, x, y)$ is the heat kernel.

Based on the exercise above, we have two equivalent ways to define the heat trace.

Definition 6. The heat trace is the trace of the heat kernel. On a compact Riemannian manifold, it is equivalently given by

$$\int_M H(t, x, x) dx = \sum_{k \geq 0} e^{-\lambda_k t},$$

where $\{\lambda_k\}_{k \geq 0}$ are the eigenvalues of the Laplacian on M.

Theorem 7. *The heat kernel has the asymptotic expansion*

$$H(t, x, x) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} u_k(x, x) t^k.$$

Proof: We have proven that $H(t, x, x) = (h_k - Q_k * h_k)(t, x, x)$ for large k. We have estimated before that $Q_k = O(t^{k-n/2})$.

Exercise 4. *Verify that because of the extra t integral together with the integral over M that $Q_k * h_k$ gains one power of t, becoming $O(t^{k+1-n/2})$.*

We therefore have

$$(4\pi t)^{n/2} h_k(t, x, x) = u_0(x, x) + t u_1(x, x) + \ldots + t^k u_k(x, x) + O(t^{k+1}).$$

Since we can do this for any k, because for large k the heat kernel is equal to $h_k - Q_k * h_k$ for all k, we obtain the expansion in the theorem.

cor:httr **Corollary 8.** *The heat kernel satisfies*

$$\int_M H(t,x,x)dx \sim (4\pi t)^{-n/2} \sum_{k\geq 0} \int_M u_k(x,x) t^k dx.$$

In particular,

$$\lim_{t\to 0} \int_M H(t,x,x)dx (4\pi t)^{n/2} = \mathrm{Vol}(M).$$

Proof: This follows immediately, together with recalling that $u_0(x,x) = 1$.

Consequently, we also have

$$\sum_{k\geq 0} e^{-\lambda_k t} \sim \frac{\mathrm{Vol}(M)}{(4\pi t)^{n/2}} \mathrm{t} \downarrow 0.$$

1.1. **Spectral invariants.** The set of eigenvalues of the Laplacian is known as the *spectrum.* Any quantity which is determined by these numbers is known as a *spectral invariant.* Since the heat trace,

$$\sum_{k\geq 0} e^{-\lambda_k t}$$

is determined by the spectrum, it is a spectral invariant. Hence, the rate at which it blows up when $t \to 0$ is also a spectral invariant. If we know this rate then we know two things: the volume of the manifold and its dimension. So, those are spectral invariants. Moreover, all of the integrals

$$\int_M u_k(x,x)dx$$

are also spectral invariants by Corollary cor:httr 8. What information do these possess? We recall how we obtained the functions $u_k(x,y)$. For y close to x, $u_0(x,y) = \det^{-1/2} g(y)$ has a Taylor expansion in the components of y whose coefficients are universal polynomials in the Riemannian curvature tensor, R_x at x, together with its covariant derivatives. This is because the metric components g_{ij} have such a Taylor expansion and $x^{-1/2}$ has a Taylor expansion as well. Assuming that u_{i-1} has such a Taylor expansion in y whose coefficients are given by universal polynomials in the curvature and its covariant derivatives, for all $x, y \in M$ which are close, then we have that

$$u_i(x,y) = -r^{-i}(x,y) \det(y)^{-\frac{1}{2}} \int_0^r \det(x(s))^{\frac{1}{2}} \Delta_u u_{i-1}(x(s)y) s^{i-1} ds.$$

Here $r = d(x,y)$, and $x(s)$ is the geodesic from x to $y = x(r)$. If $y = (y^1, \ldots, y^n)$ in Riemannian normal coordinates at x, then $r^2(y) = g_{ij} y^i y^j$, so $r^{-i}(y)$ has a Taylor expansion in y with coefficients given by universal polynomials in the curvature and its covariant derivatives. In general, these polynomials are super complicated. Nonetheless, the integrals of these polynomials,

$$\int_M u_k(x,x)dx,$$

are spectral invariants which can be computed if we can determine the coefficients of the powers of t in the asymptotic expansion of the heat trace as $t \downarrow 0$. In this way, we see that curvature and heat are intertwined.

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1.2. **Weyl's Law via the Heat Trace and Karamata's Lemma.** As an interesting application of our results for the heat kernel and its trace, we shall determine the *rate* at which the eigenvalues $\lambda_k \to \infty$ when $k \to \infty$. This is known as *Weyl's Law.* In general, there are two main techniques which can be used to prove Weyl's law. The first technique is known as *Dirichlet-Neumann bracketing,* and this is the one used in Weyl's classical proof. The second technique uses an element of the functional calculus of the Laplacian, such as the resolvent, wave group, or heat semi-group, together with a suitable Tauberian theorem. Here, we use the short time asymptotic behavior of the heat trace together with Karamata's Tauberian Lemma.

klemma2

Lemma 9. *Let g be a bounded, non-negative, piecewise continuous function on $[0,1]$. Assume that ν is a non-negative measure, $\alpha \geq 1$, and*

$$\int_0^\infty e^{-t\lambda} d\nu(\lambda) < \infty \, \forall t > 0, \quad \lim_{t \downarrow 0} t^\alpha \int_0^\infty e^{-t\lambda} d\nu(\lambda) = c \in (0, \infty).$$

Then, we have

$$\lim_{t \downarrow 0} t^\alpha \int_0^\infty g(e^{-t\lambda}) e^{-t\lambda} d\nu(\lambda) = \frac{c}{\Gamma(\alpha)} \int_0^\infty g(e^{-t}) e^{-t} t^{\alpha-1} dt.$$

Proof: We begin by introducing the notations

$$F_t(g) = t^\alpha \int_0^\infty g\left(e^{-t\lambda}\right) e^{-t\lambda} \mathrm{d}\nu(\lambda), \quad G(g) = \frac{c}{\Gamma(\alpha)} \int_0^\infty g\left(e^{-t}\right) t^{\alpha-1} e^{-t} \mathrm{d}t. \tag{1.1}$$

Let $\{g_n^L\}, \ \{g_n^U\}$ be two sequences of continuous functions such that

$$0 \leq g_1^L(x) \leq \ldots \leq g_n^L(x) \leq \ldots \leq g(x) \leq \ldots \leq g_n^U(x) \ldots \leq g_1^U(x) \leq M, \ \text{ for a.e. } x \in [0,1],$$

and

$$\lim_{n \to \infty} g_n^L(x) = \lim_{n \to \infty} g_n^U(x) = g(x) \quad \text{ for a.e. } x \in [0,1].$$

Due to the assumption that g is non-negative and bounded, we have the estimate

$$0 \leq G(g) \leq \frac{c}{\Gamma(\alpha)} \int_0^\infty M t^{\alpha-1} e^{-t} dt = cM.$$

This will allow us to apply dominated convergence arguments.

Since all of the g_n^L and g_n^U are continuous, by Karamata's original Lemma,

$$\lim_{t \downarrow 0} F_t(g_n^L) = G(g_n^L) \ \text{ and } \ \lim_{t \downarrow 0} F_t(g_n^U) = G(g_n^U)$$

for each n. Therefore,

$$\limsup_{t \downarrow 0} F_t(g) \leq \limsup_{t \downarrow 0} F_t(g_n^U) = G(g_n^U)$$

and

$$\liminf_{t \downarrow 0} F_t(g) \geq \liminf_{t \downarrow 0} F_t(g_n^L) = G(g_n^L).$$

Together these estimates give

$$G(g_n^L) \leq \liminf_{t \downarrow 0} F_t(g) \leq \limsup_{t \downarrow 0} F_t(g) \leq G(g_n^U)$$

for all n.

By dominated convergence we have

$$\lim_{n \to \infty} G(g_n^L) = G(g) = \lim_{n \to \infty} G(g_n^U)$$

and as a result

$$\lim_{t \downarrow 0} F_t(g) = G(g).$$

Theorem 10 (Weyl's law for the eigenvalues of the Laplacian on a Riemannian manifold)**.** *We assume that (M, g) is a compact Riemannian manifold. Let $\{\lambda_k\}_{k \geq 1}$ be the eigenvalues of the Laplace operator. Let*

$$N(\Lambda) := \#\{\lambda_k \leq \Lambda\}.$$

Then

$$\lim_{\Lambda \to \infty} \frac{N(\Lambda) n \Gamma\left(\frac{n}{2}\right) (4\pi)^{n/2}}{2 \Lambda^{n/2} \operatorname{Vol}(M)} = 1.$$

Proof: We shall apply the generalized Karamata Lemma to the measure

$$d\nu := \sum_{k \geq 1} \delta_{\lambda_k}.$$

We shall apply Lemma klemma2 9 to the function,

cleverf

$$g(x) = \begin{cases} 0; & x \in \left[0, e^{-1}\right] \cup [1, \infty) \\ \frac{1}{x}; & x \in \left(e^{-1}, 1\right) \end{cases} . \tag{1.2}$$

By Corollary cor:httr 8

$$\lim_{t \downarrow 0} t^{n/2} \int_0^\infty e^{-t\lambda} d\nu(\lambda) = \lim_{t \downarrow 0} t^{n/2} \left(\sum_{k \geq 0} e^{-\lambda_k t} \right) = \frac{\operatorname{Vol}(M)}{(4\pi)^{n/2}}.$$

Let

$$c := \frac{\operatorname{Vol}(M)}{(4\pi)^{n/2}}, \quad \alpha = \frac{n}{2}$$

Applying Lemma klemma2 9 with the aforementioned g, c, and α we obtain

$$\lim_{t \downarrow 0} t^\alpha \int_0^\infty g\left(e^{-t\lambda}\right) e^{-t\lambda} \mathrm{d}\nu(\lambda) = \frac{c}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty g\left(e^{-t}\right) t^{\frac{n}{2}-1} e^{-t} \mathrm{d}t.$$

Substituting for g in this equation we have

$$\lim_{t \downarrow 0} t^\alpha \int_0^{1/t} e^{t\lambda} e^{-t\lambda} \mathrm{d}\nu(\lambda) = \lim_{t \downarrow 0} t^\alpha N\left(\frac{1}{t}\right) = \frac{c}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 e^t t^{n/2-1} e^{-t} \mathrm{d}t = \frac{c}{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)}.$$

Clearly then,

$$\lim_{t \to 0} t^{n/2} N\left(\frac{1}{t}\right) = \frac{2c}{n\Gamma(n/2)} \iff N(\lambda) \sim \frac{2\lambda^{n/2}}{n\Gamma\left(\frac{n}{2}\right)} \frac{\operatorname{Vol}(M)}{(4\pi)^{n/2}}, \quad \lambda \to \infty.$$

Exercise 5. *Use Weyl's law to show that*

$$\lambda_k \approx k^{2/n} \text{ as } k \to \infty,$$

where the notation means there are positive constants such that λ_k is bounded above and below by constants times $k^{2/n}$.

References

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