HEAT AND GEOMETRY

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1. Lecture Notes 2018.07.12

Imagine that a bounded, connected domain in $\mathbb{R}^2$ is made out of some flexible, vegan material and attached to a rigid body. It is the head of a drum. In this situation, the edges of the domain are not able to vibrate, but the interior does vibrate if we beat on the drum. It turns out that the eigenvalues of the Laplacian on the domain, let us call it Ω, with the Dirichlet boundary condition, determine the frequencies produced by the vibration of the drum.

Exercise 1. *Use the eigenvalues and eigenfunctions to solve the initial value problem for the wave equation on the domain* Ω*:*

$$\begin{cases} \Box u(t,x) = 0 & x \in \Omega, t > 0 \\ u_t(0,x) = 0 & x \in \Omega \\ u(0,x) = f(x) & x \in \Omega \end{cases}$$

where above $\Box = \partial_t^2 + \Delta$*.*

1.1. Hearing the shape of a drum.

If we have got really great hearing, then we can listen to the vibrations of Ω and "hear" all of the eigenvalues $\{\lambda_k\}$. Recall that the collection of all of these is known as the spectrum. Kac's famous question is: if Ω and Ω' sound identical, in the sense that they have the exact same (counting multiplicity) set of eigenvalues, then are the domains the same shape? Mathematically, if two domains are isospectral (that is they have the exact same spectrum) then are they the same shape?

Exercise 2. *Prove that the spectrum is independent under rigid motions of the plane (translations, reflections, rotations, etc). In this way prove that it is the shape of* Ω*, not the way or place in which* Ω *sits in* $\mathbb{R}^2$ *which determines the spectrum.*

Exercise 3. *Assume that* Ω *and* Ω' *as above are both rectangles. Prove that if they are isospectral then they're the same rectangle.*

Inspired by Kac's terminology, it is common to use the expression "one can hear X" if X is a spectral invariant. Let's now use the locality principle to see what we can hear...

1.2. Locality principle.

Let's make this notion of a locality principle for domains in $\mathbb{R}^n$ precise. Let Ω be a domain, possibly infinite, contained in $\mathbb{R}^n$.

exactmatch **Definition 1.** Assume that $\Omega_0 \subset \Omega \subset \mathbb{R}^n$, and $S \subset \mathbb{R}^n$. We say that S and Ω are *exact geometric matches on* Ω_0 if there exists a sub-domain $\Omega_c \subseteq \Omega$ which compactly contains Ω_0 and which is isometric to a sub-domain of S (which, abusing notation, we also call Ω_c). Recall that Ω_0 being compactly contained in Ω_c means that the distance from $\overline{\Omega}_0$ to $\overline{\Omega \setminus \Omega_c}$ is positive. A planar example is depicted in Figure 1.

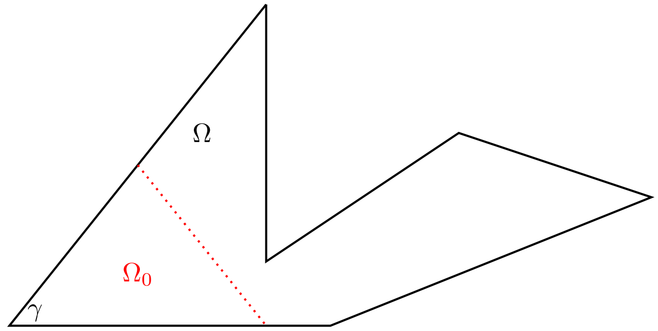

FIGURE 1. Above, we have the polygonal domain Ω which contains the triangular domain, Ω_0. Letting $S = S_\gamma$ be a circular sector of opening angle γ and infinite radius, this is an example of an "exact geometric match," in the sense that Ω_0 is equal to a piece of S.

fig2

In the case of smoothly bounded domains, Lück & Schick [6, Theorem 2.26], implies the locality principle for both the Dirichlet and Neumann boundary conditions, and which holds all the way up to the boundary. We recall that result.[1]

Theorem 2 (Lück & Schick). *Let N be a Riemannian manifold possibly with boundary which is of bounded geometry. Let $V \subset N$ be a closed subset which carries the structure of a Riemannian manifold of the same dimension as N such that the inclusion of V into N is a smooth map respecting the Riemannian metrics. For fixed $p \geq 0$, let $\Delta[V]$ and $\Delta[N]$ be the Laplacians on p-forms on V and N, considered as unbounded operators with either absolute boundary conditions or with relative boundary conditions (see Definition 2.2 of [6]). Let $\Delta[V]^k e^{-t\Delta[V]}(x,y)$ and $\Delta[N]^k e^{-t\Delta[N]}(x,y)$ be the corresponding smooth integral kernels. Let k be a non-negative integer.*

Then there is a monotone decreasing function $C_k(K) : (0,\infty) \to (0,\infty)$ which depends only on the geometry of N (but not on V, x, y, t) and a constant C_2 depending only on the dimension of N such that for all $K > 0$ and $x, y \in V$ with $d_V(x) := d(x, N \setminus V) \geq K$, $d_V(y) \geq K$ and all $t > 0$:

$$\left| \Delta[V]^k e^{-t\Delta[V]}(x,y) - \Delta[N]^k e^{-t\Delta[N]}(x,y) \right| \leq C_k(K) e^{-\left(\frac{d_V(x)^2 + d_V(y)^2 + d(x,y)^2}{C_2 t} \right)}.$$

The locality principle for functions corresponds to the Laplacian on p-forms for $p = 0$. It is therefore obtained as a corollary to Lück & Schick's result.

Corollary 3. *Assume that S is an exact match for $\Omega_0 \subset \Omega$, for two smoothly bounded domains, Ω and Ω_0 in $\mathbb{R}^n$. Assume the same boundary condition, either Dirichlet or Neumann, for the Euclidean Laplacian on both domains. Then*

$$\left| H^\Omega(t,z,z') - H^S(t,z,z') \right| = O(t^\infty) \text{ as } t \downarrow 0, \quad \text{uniformly for } z, z' \in \Omega_0.$$

[1] In the original statement of their result, Lück and Schick make the parenthetical remark "We make no assumptions about the boundaries of N and V and how they intersect." This could easily be misunderstood. If one carefully reads the proof, it is implicit that the boundaries are *smooth*. The arguments break down if the boundaries have singularities, such as corners. For this reason, we have omitted the parenthetical remark from the statement of the theorem.

Proof: We use the theorem of Lück and Schick twice, once with $N = \Omega$ and once with $N = S$, with $V = \Omega_c$ in both cases. We set $k = 0$ and

$$K = \alpha = d(\Omega_0, S \setminus \Omega_c).$$

By the definition of an exact geometric match, $\alpha > 0$. In the $N = S$ case, the theorem reads

$$|H^S(t,z,z') - H^{\Omega_c}(t,z,z')| \leq C_0(\alpha) e^{-\frac{|\mathrm{dist}(z, S\setminus\Omega_c)|^2}{C_2 t} - \frac{|\mathrm{dist}(z', S\setminus\Omega_c)|^2}{C_2 t}} \leq C_0(\alpha) e^{-\frac{2\alpha^2}{C_2 t}}.$$

We conclude that

$$|H^S(t,z,z') - H^{\Omega_c}(t,z,z')| = O(t^\infty)$$

uniformly on Ω_0. The same statement holds with S replaced by Ω, and then the triangle inequality completes the proof.

The locality principle tells us that if we have some piece of a domain which is equal to some piece of a domain for which we know the heat kernel, then we can use that known heat kernel to compute for our original domain. In particular, if we wish to compute the asymptotics of the heat trace as $t \downarrow 0$, we simply integrate of patches of a domain using model heat kernels for each patch, and then sum up the results. In [10] we proved a locality principle which also holds when the boundary is not necessarily smooth.

Definition 4. Let $\epsilon > 0$ and $h > 0$. We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies the (ε, h)-cone condition if, for every $x \in \partial\Omega$, there exists a ball $B(x, \delta)$ centered at x of radius δ, and a direction ξ_x, such that for all $y \in B(x,\delta) \cap \Omega$, the cone with vertex y directed by ξ_x of opening angle ε and height h is contained in Ω.

Definition 5. Let $\epsilon > 0$ and $h > 0$. We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies the two-sided (ε, h)-cone condition if both Ω and $\mathbb{R}^n \setminus \Omega$ satisfy the (ε, h)-cone condition.

The following locality principle was proven for the Neumann boundary condition in [10]. If you're curious about the proof please see [10] which is available online!

Theorem 6. *Let Ω, Ω_0, and S be domains in $\mathbb{R}^n$ such that S and Ω are exact geometric matches on Ω_0, as in Definition 1. Assume that both Ω and S satisfy the two-sided (ε, h)-cone condition for some $\varepsilon > 0$ and $h > 0$. Let H^Ω denote the heat kernel associated to the Laplacian on Ω, and let H^S denote the heat kernel on S, with the same boundary condition for ∂S as taken on $\partial\Omega$. Moreover, assume that there exists $\sigma \in \mathbb{R}$ such that the second fundamental form $II \geq -\sigma$ holds on all the $\mathcal{C}^2$ pieces of $\partial\Omega$ and ∂S. Then*

$$\left|H^\Omega(t,z,z') - H^S(t,z,z')\right| = O(t^\infty) \text{ as } t \downarrow 0, \quad \text{uniformly for } z, z' \in \Omega_0.$$

The theorem was proven in [10] for the Neumann and Robin boundary conditions, but it is straightforward (and in fact easier) to obtain it for the Dirichlet boundary condition. This modification of the proof is left as an **exercise.**

1.3. **Polygonal domains.** Let us consider a polygonal domain. Let's assume, like Kac, the Dirichlet boundary condition. We shall use the locality principle to compute the short time asymptotic expansion of the heat trace by computing the contributions to the trace from three local models:

(1) the heat kernel for $\mathbb{R}^2$ in the interior

(2) the heat kernel for a half space near the edges but away from the corners
(3) the heat kernel for an infinite circular sector near the corners

Integrating the Euclidean heat kernel at the diagonal we just get

$$\text{area over which we integrate } 4\pi t.$$

Away from the corners of the polygon, a piece of a half-space is an exact match for the domain. What's the heat kernel for a half-space with the Dirichlet boundary condition? A half space is the product

$$[0, \infty) \times \mathbb{R}.$$

Exercise 4. *Let M and N be two domains or Riemannian manifolds, such that they have well-defined Laplacians and corresponding heat kernels. Define the product space $M \times N$. Then the heat kernel on the product space is equal to*

$$H_M(t, x, y) H_N(t, x, y),$$

where H_M and H_N are the heat kernels on M and N, respectively.

The heat kernel for a half line with the Dirichlet boundary condition at the origin is

$$H(t, x, x') = e^{-|x-x'|^2/(4t)}(4\pi t)^{-1/2} - e^{-|x+x'|^2/(4t)}(4\pi t)^{-1/2}.$$

Coincidentally, we can obtain this formula with the help of an old Scottish-Australian mathematician, Carslaw, and his "method of images." We know the heat kernel for $\mathbb{R}$, so we can use the exercise (teamwork!) to obtain the heat kernel for the half space:

$$\{z = (x, y) \in \mathbb{R} : y \geq 0\}$$

is

$$H(t, z, z') = (4\pi t)^{-1/2} e^{-(x-x')^2/(4t)} \left(e^{-|y-y'|^2/(4t)}(4\pi t)^{-1/2} - e^{-|y+y'|^2/(4t)}(4\pi t)^{-1/2} \right).$$

Let's compute the integral along the diagonal, $z = z'$ near the boundary. So, this is

$$(4\pi t)^{-1} \int_{x=a}^{x=b} \int_{y=0}^{\varepsilon} \left(1 - e^{-y^2/4t}\right) dx dy = \frac{b-a}{4\pi t} \left(\varepsilon - \int_0^{\varepsilon} e^{-y^2/4t} dy \right).$$

Changing variables by letting $s = yt^{-1/2}$, so that $\sqrt{t} ds = dy$, the integral becomes

$$\sqrt{t} \int_0^{\varepsilon/\sqrt{t}} e^{-s^2} ds = \sqrt{t} \left(\int_0^{\infty} e^{-s^2} ds - \int_{\varepsilon/\sqrt{t}}^{\infty} e^{-s^2} ds \right)$$
$$= \sqrt{t} \frac{\sqrt{\pi}}{2} - \frac{\sqrt{t\pi}}{2} \operatorname{erfc}(\varepsilon t^{-1/2}).$$

Above erfc is the complementary error function. We have therefore computed that the integral of the heat kernel for the half space along the diagonal near the boundary of the half space is

$$\frac{(b-a)\varepsilon}{4\pi t} - \frac{(b-a)}{8\sqrt{\pi t}} - \frac{(b-a)}{8\sqrt{\pi t}} \operatorname{erfc}(\varepsilon t^{-1/2}).$$

Geometrically, this is

$$\frac{\text{area}}{4\pi t} - \frac{\text{perimeter}}{8\sqrt{\pi t}} - \frac{\text{perimeter}}{8\sqrt{\pi t}} \operatorname{erfc}(\varepsilon t^{-1/2}).$$

So, all that we need to understand now is what happens near the corners. Near a corner of opening angle γ, an infinite sector with the same opening angle is an exact geometric match. In [?nrs2] we show how to compute the heat kernel for such sectors, for pretty much any boundary condition. The case of Dirichlet boundary condition goes back to Boris Fedosov [2]. The heat kernel is obtained by taking the inverse Laplace transform of the Green's function, which is:

DirichletGK
$$G_D(s,r,\phi,r_0,\phi_0) = \frac{1}{\pi^2}\int_0^\infty K_{i\mu}(rs)K_{i\mu}(r_0 s) \tag{1.1}$$

$$\times\left\{\cosh(\pi-|\phi_0-\phi|)\mu - \frac{\sinh\pi\mu}{\sinh\gamma\mu}\cosh(\phi+\phi_0-\gamma)\mu + \frac{\sinh(\pi-\gamma)\mu}{\sinh\gamma\mu}\cosh(\phi-\phi_0)\mu\right\}d\mu.$$

Please excuse the formality, but let's just see briefly the motivation behind this. The heat kernel is the integral kernel of the operator $e^{-t\Delta}$. The Green's function is the integral kernel of the resolvent operator $(\Delta+s)^{-1}$, where the spectral parameter is s. Then, if we take the Laplace transform:

$$\mathcal{L}(e^{-t\Delta})(s) = \int_0^\infty e^{-ts}e^{-t\Delta}dt = \int_0^\infty e^{-t(s+\Delta)}dt = (s+\Delta)^{-1}.$$

This is purely formal, basically we are playing calculus with pseudodifferential operators. It turns out that this is actually rigorously justified using techniques known as *functional calculus.* Great contributions to functional calculus were made by the Australian mathematician, Alan McIntosh. So, thanks to Alan and Boris we have an explicit, albeit somewhat complicated, expression for the heat kernel. Let

$$A := \int_0^\infty K_{i\mu}(rs)K_{i\mu}(r_0 s)\cosh(\pi-|\phi_0-\phi|)\mu d\mu,$$

$$B := \int_0^\infty K_{i\mu}(rs)K_{i\mu}(r_0 s)\frac{\sinh\pi\mu}{\sinh\gamma\mu}\cosh(\phi+\phi_0-\gamma)\mu d\mu$$

$$C := \int_0^\infty K_{i\mu}(rs)K_{i\mu}(r_0 s)\frac{\sinh(\pi-\gamma)\mu}{\sinh\gamma\mu}\cosh(\phi-\phi_0)\mu d\mu.$$

Then the Dirichlet Green's function is

$$\frac{1}{\pi^2}(A-B+C).$$

1.3.1. *Heat trace contribution from the A term.* Setting $\phi=\phi_0$, we have by [3, 6.794.1]

$$\int_0^\infty K_{ix}(r\sqrt{s})K_{ix}(r_0\sqrt{s})\cosh(\pi x)dx = \frac{\pi}{2}K_0(\sqrt{(r-r_0)^2 s}).$$

Then, by [1, 5.16.35], we have

$$\mathcal{L}^{-1}[A] = \mathcal{L}^{-1}\left[\frac{\pi}{2}K_0(\sqrt{(r-r_0)^2 s})\right] = \frac{\pi}{2}\frac{1}{2}\frac{1}{t}e^{-\frac{(r-r_0)^2}{4t}}.$$

Hence for $\phi=\phi_0$,

A-phi
$$\frac{1}{\pi^2}\mathcal{L}^{-1}(A) = \frac{e^{-\frac{(r-r_0)^2}{4t}}}{4\pi t}. \tag{1.2}$$

Setting $r=r_0$ gives $(4\pi t)^{-1}$, and integrating over $\mathcal{N}_i$, the contribution from this term to the heat trace is the usual area term:

$$\frac{A(\mathcal{N}_i)}{4\pi t}.$$

1.3.2. *Heat trace contribution from the B term.* Now we investigate the contribution from B. The first simplification is to restrict to $\phi = \phi_0$, then compute

$$\int_0^\gamma B|_{\phi=\phi_0} d\phi = \int_0^\gamma K_{ix}(rs)K_{ix}(r_0 s)\frac{\sinh \pi x}{\sinh \gamma x}\cosh(2\phi - \gamma)x d\phi.$$

The only dependence on the angle is in the cosh term, which may be explicitly integrated, and we obtain

$$\int_0^\gamma B|_{\phi=\phi_0} d\phi = \int_0^\infty K_{ix}(r\sqrt{s})K_{ix}(r_0\sqrt{s})\frac{\sinh \pi x}{x}dx = \frac{\pi^2}{2}I_0(r_0\sqrt{s})K_0(r\sqrt{s}),$$

where in the last equality we have used [gr 3, 6.794.10]. Now take the inverse Laplace transform:

$$\mathcal{L}^{-1}\left[\int_0^\gamma B|_{\phi=\phi_0} d\phi\right] = \mathcal{L}^{-1}\left[\frac{\pi^2}{2}I_0(r_0\sqrt{s})K_0(r\sqrt{s})\right] = \frac{\pi^2}{2}\frac{1}{2t}e^{-\frac{r^2+r_0^2}{4t}}I_0(\frac{rr_0}{2t}). \tag{1.3}$$

Thus, we see that

Bphi
$$\frac{1}{\pi^2}\mathcal{L}^{-1}\left[\int_0^\gamma B|_{\phi=\phi_0} d\phi\right] = \frac{1}{4t}e^{-\frac{r^2+r_0^2}{4t}}I_0(\frac{rr_0}{2t}). \tag{1.4}$$

To compute the trace, we make a change of variables, by setting

$$u = \frac{r^2}{2t}, \quad du = \frac{r}{t}dr.$$

Therefore,

$$\frac{1}{4t}\int_0^R e^{-r^2/2t}I_0\left(\frac{r^2}{2t}\right)rdr = \frac{1}{4}\int_0^{\frac{R^2}{2t}} e^{-u}I_0(u)du.$$

By [watson 13, p. 79 (3)] with $\nu = 1$,

wa79-3
$$uI_1'(u) + I_1(u) = uI_0(u). \tag{1.5}$$

By [watson 13, p. 79 (4)] with $\nu = 0$,

wa79-4
$$uI_0'(u) = uI_1(u). \tag{1.6}$$

We use these to compute

$$\begin{aligned}\frac{d}{du}\left(e^{-u}u(I_0(u)+I_1(u))\right) &= e^{-u}\left(-uI_0(u) - uI_1(u) + I_0(u) + I_1(u) + uI_0'(u) + uI_1'(u)\right)\\ &= e^{-u}\left(-uI_1(u) + I_0(u) + uI_0'(u)\right) \quad (\text{by (1.5)})\\ &= e^{-u}I_0(u) \quad (\text{by (1.6)}).\end{aligned}$$

Next, define

magic
$$g(u) := e^{-u}u(I_0(u) + I_1(u)), \tag{1.7}$$

and note that we have computed

$$g'(u) = e^{-u}I_0(u).$$

We therefore have

$$\int_0^{R^2/2t} e^{-u}I_0(u)du = \left(g(R^2/2t) - g(0)\right).$$

Since $I_0(0) = 1$ and $I_1(0) = 0$ [13], it follows that $g(0) = 0$, and we therefore compute that

$$\int_0^{R^2/2t} e^{-u} I_0(u)du = g(R^2/2t) = e^{-R^2/2t}\frac{R^2}{2t}(I_0(R^2/2t) + I_1(R^2/2t)).$$

For large arguments, the Bessel functions admit the following asymptotic expansions (see [13])

$$I_j(x) = \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{1}{2x}\left(j^2 - \frac{1}{4}\right) + \sum_{k=2}^{\infty} c_{j,k}x^{-k}\right), \quad x \gg 0, \quad j = 0, 1.$$

Consequently, for $x = R^2/2t$,

$$g(R^2/2t) = \frac{R^2}{2t}e^{-R^2/2t}(I_0(R^2/2t)+I_1(R^2/2t)) = \frac{R^2}{2t}\left(\frac{2}{\sqrt{2\pi(R^2/2t)}}\right)-O\left(\frac{1}{(R^2/2t)^{3/2}}\right)$$

$$= \frac{R}{\sqrt{\pi t}} + O(\sqrt{t}), \quad t \downarrow 0.$$

Recalling the factor of $\frac{1}{4}$, we see that the trace of B contributes

$$\frac{R}{4\sqrt{\pi t}} + O(\sqrt{t}), \quad t \downarrow 0. \tag{1.8}$$

Observe that this is precisely the usual perimeter term:

$$\frac{\ell(\mathcal{N}_i \cap \partial\Omega)}{8\sqrt{\pi t}} + O(\sqrt{t}).$$

1.3.3. *Heat trace contribution from the C term.* Next, we compute the trace of the C term. This is done following [12]. The cosh term drops out when $\phi = \phi_0$. Integrating with respect to the angle gives a factor of γ. We define

$$R(t) = -\mathcal{L}^{-1}\left(\frac{\gamma}{\pi^2}\int_0^\infty \frac{\sinh(\pi-\gamma)x}{\sinh(\gamma x)}\int_R^\infty K_{ix}^2(r\sqrt{s})rdr\right).$$

It is shown in [12] that

$$R(t) = O(e^{-c/t}),$$

and in fact an estimate is also obtained there for the constant $c > 0$. Hence, it suffices to compute

$$\mathcal{L}^{-1}\left(\frac{\gamma}{\pi^2}\int_0^\infty dx\frac{\sinh(\pi-\gamma)x}{\sinh\gamma x}\int_0^\infty K_{ix}^2(rs)rdr\right).$$

Here we use [3, 6.521.3]. As in that notation we have $a = s = b$, we must compute instead the limit of the expression as $b \to a$,

$$\lim_{b\to a}\frac{\pi(ab)^{-\nu}(a^\nu + b^\nu)}{2\sin(\nu\pi)(a+b)}\frac{f(a)-f(b)}{a-b}, \quad f(t) = t^\nu.$$

Then, since

$$f'(t) = \nu t^{\nu-1}$$

we have

$$\lim_{b\to a}\frac{\pi(ab)^{-\nu}(a^\nu + b^\nu)}{2\sin(\nu\pi)(a+b)}\frac{f(a)-f(b)}{a-b} = \frac{\pi a^{-2\nu}(2a^\nu)}{4\sin(\nu\pi)a}\nu a^{\nu-1} = \frac{\pi\nu}{2\sin(\pi\nu)a^2}.$$

Inserting our parameters, we have that

$$\int_0^\infty K_{ix}^2(r\sqrt{s})rdr = \frac{\pi x}{2\sinh(\pi x)s}.$$

So we must compute

$$\mathcal{L}^{-1}\left\{\frac{\gamma}{\pi^2}\int_0^\infty \frac{\sinh(\pi-\gamma)x}{\sinh\gamma x}\frac{\pi x}{2s\sinh(\pi x)}dx\right\}.$$

This calculation has been done in [12, p. 122] using [3]; we have independently verified these calculations as well. The result is given in [12, (2.10)]:

$$\frac{\pi^2-\gamma^2}{24\pi\gamma}.$$

Thus, we see that C contributes to the trace the usual "corner contribution":

$$\frac{\pi^2-\gamma^2}{24\pi\gamma} + O(t^\infty). \tag{1.9}$$

Exercise 5. *Use the reference [3] together with [12] to verify this calculation. (It's not as easy as it sounds because van den Berg and Srisatkunarajah do some really clever tricks).*

So, we have computed that all together the trace of the heat kernel on a convex polygonal domain

$$\mathrm{tr} e^{-t\Delta} \sim \frac{\text{Area}}{4\pi t} - \frac{\text{perimeter}}{8\sqrt{\pi t}} + \sum_{i=1}^n \frac{\pi^2-\gamma_i^2}{24\pi\gamma_i} + O(t^\infty).$$

This shows that the area, perimeter and that funny expression involving the angles are all spectral invariants.

1.4. **Kac's holes.** Now, let us do what Kac did next. We wish to use our polygonal domain to approximate a smoothly bounded domain. So, to do this, we use lots and lots of angles which shall tend towards π. For the sake of simplicity, let us consider a convex domain. Let's take the angles of the polygon we use to approximate the smooth domain to be all equal. Since the domain is convex, this means that for N angles, they are each

$$\frac{\pi(N-2)}{N}.$$

The first two terms in the heat trace are as above, in terms of area and perimeter, whereas the last one is

$$\frac{1}{24}\left(\frac{N^2}{N-2} - (N-2)\right) = \frac{1}{24}\left(\frac{N^2-(N-2)^2}{N-2}\right) = \frac{1}{24}\left(\frac{4N-4}{N-2}\right).$$

We therefore compute the limit as $N \to \infty$ to be

$$\frac{4}{24} = \frac{1}{6}.$$

So, this is what we get corresponding to going around the smooth boundary on the outside of the domain.

What if our smoothly bounded domain also has some holes inside? In that case, note that if the angle inside a corner is γ, then the surrounding outside angle is $2\pi - \gamma$. So we compute using angles $2\pi - \frac{\pi(N-2)}{N}$:

$$\frac{1}{24}\sum_{i=1}^{N} \frac{\pi^2 - (2\pi - \gamma_i)^2}{\pi(2\pi - \gamma_i)},$$

and then setting $\gamma_i = \frac{\pi(N-2)}{N}$ for all i this is

$$\frac{1}{24}\left(N\frac{\pi}{2\pi - \pi(N-2)/N} - N\frac{(2\pi - \pi(N-2)/N}{\pi}\right)$$

$$= \frac{1}{24}\left(\frac{N^2}{2N-(N-2)} - (2N-(N-2))\right) = \frac{1}{24}\left(\frac{N^2}{N+2} - (N+2)\right),$$

so

$$\lim_{N\to\infty} \frac{1}{24}\frac{N^2-(N+2)^2}{N+2} = -\frac{4}{24} = -\frac{1}{6}.$$

Hence, each time there is a hole inside the domain, we get a contribution of $-\frac{1}{6}$. Kac therefore concluded that for a domain with h holes, the heat trace is asymptotic to

$$\frac{\text{area}}{4\pi t} - \frac{\text{perimeter}}{8\sqrt{\pi t}} + \frac{1-h}{6}, \text{ h is the number of holes in the domain..}$$

Now, there is a "hole" in this argument, because we are basically swapping limits around. Kac acknowledged this which is pretty awesome. It turns out that, interestingly, the calculation works. This was rigorously justified in the first paper I asked y'all to read, [8].

Let's conclude with an exercise.

Exercise 6. *Let Ω be a smoothly bounded domain in $\mathbb{R}^2$. Let P be a convex polygonal domain in $\mathbb{R}^2$. Assume the Dirichlet boundary condition for both domains. Prove that Ω and P cannot be isospectral. In this sense we say that "One can hear the corners of a drum" (if you get stuck, check out [5]).*

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