HEAT AND GEOMETRY

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1. 2018.07.09 Pre-Lecture Notes

The purpose of the pre-lecture notes is to prepare you for the forthcoming lecture. It's a pedagogically known fact that taking your own notes during the actual lecture is beneficial to learning and memory. For that reason, everyone is encouraged to take notes during the lecture. After the lecture, a second set of notes will be made available. It is advisable to review these notes together with your own lecture notes to solidify your understanding of the material. Moreover, there are several **exercises** throughout the post-lecture notes which are recommended.

1.1. Summary of the 2018.07.09 lecture.

We shall:

(1) Define the heat operator and Laplace operator on a Riemannian manifold.
(2) Study the initial value problem for the heat equation on $\mathbb{R}^n$.
(3) In this context, we will use the Fourier transform and convolutions, together with some basics about $\mathcal{L}^1$ and $\mathcal{L}^2$.
(4) We'll use these tools to solve the IVP for the heat equation on $\mathbb{R}^n$ and thereby obtain the Euclidean heat kernel.
(5) We shall make some elementary observations about this heat kernel.
(6) The heat kernel is for the most part well-behaved, but it has a certain discontinuity. To resolve this, we will introduce the heat space.
(7) In this context, we will define manifolds with corners, p-submanifolds, standard blowups and parabolic blowups.

1.2. Warm-up exercises.

To get warmed up for the first lecture and this lecture series in general, here are a few fun exercises:

(1) Write up your own explanation of the physical derivation of the heat equation based on the laws of Thermodynamics. It may be simplest to do this first in one dimension and then to explain how the derivation generalizes to all dimensions. If you're curious and bored, read about Joseph Fourier. Apparently he was somewhat of a character...
(2) Track down the first ever published "Journal of Differential Geometry." Look for an article called *Curvature and eigenvalues of the Laplacian.*
(3) Try to wrap your head around the notion of blowing up. See for example [2, p. 252–259] and [1, §2.1–2.2]. There are other references as well; see what you can find!

convolutionapprox

1.3. Supplementary material: convolution approximation theorem in one dimension.

This theorem shall be related to the fact that the heat kernel gives the solution to the heat equation which converges as $t \downarrow 0$ to the initial data. The proof of the analogous result in $\mathbb{R}^n$ shall be an exercise in the mini-course. Feel free to skip this if it's too basic for ya, otherwise, feel free to use this as a bit of a warm-up as well as to get the main ideas for the general case of $\mathbb{R}^n$.

Theorem 1. *Let $g \in L^1(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} g(x)dx = 1.$$

Define

$$\alpha = \int_{-\infty}^{0} g(x)dx, \quad \beta = \int_{0}^{\infty} g(x)dx.$$

Assume that f is piecewise continuous on $\mathbb{R}$ and its left and right sided limits exist for all points of $\mathbb{R}$. Assume that either f is bounded on $\mathbb{R}$ or that g vanishes outside of a bounded interval. Let, for $\varepsilon > 0$,

$$g_\epsilon(x) = \frac{g(x/\epsilon)}{\epsilon}.$$

Then

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \forall x \in \mathbb{R}.$$

Proof. We would like to show that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y) dy = \alpha f(x+) + \beta f(x-)$$

which is equivalent to showing that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y) dy - \alpha f(x+) - \beta f(x-) = 0.$$

We now insert the definitions of α and β, so we want to show that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^{0} f(x+) g(y) dy - \int_{0}^{\infty} f(x-) g(y) dy = 0.$$

We can prove this if we show that

$$\lim_{\varepsilon \to 0} \int_{-\infty} f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^{0} f(x+) g(y) dy = 0$$

and also

$$\lim_{\varepsilon \to 0} \int_{0}^{\infty} f(x-y) g_\varepsilon(y) dy - \int_{0}^{\infty} f(x-) g(y) dy = 0.$$

The argument is the same for both of these, so proving one of them is sufficient. We choose the first.

Hence, we would like to show that by choosing ε sufficiently small, we can make

$$\int_{-\infty}^{0} f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^{0} f(x+) g(y) dy$$

as small as we like. To make this precise, let us assume that "as small as we like" is quantified by a very small $\delta > 0$. We smash the two integrals together, writing

$$\int_{-\infty}^{0} \left(f(x-y) g_\varepsilon(y) - f(x+) g(y) \right) dy.$$

Well, this is a bit inconvenient, because in the first part we have g_ε, but in the second part it's just g. So, we make a small observation,

$$\int_{-\infty}^0 g(y)dy = \int_{-\infty}^0 g(z/\varepsilon)\frac{dz}{\varepsilon} = \int_{-\infty}^0 g_\varepsilon(z)dz$$

Above, we have made the substitution $z = \varepsilon y$, so $y = z/\varepsilon$, and $dz/\varepsilon = dy$. The limits of integration don't change. By this calculation,

$$\int_{-\infty}^0 f(x+)g(y)dy = \int_{-\infty}^0 f(x+)g_\varepsilon(y)dy.$$

Note that $f(x+)$ is a constant, so it's just sitting there doing nothing. Hence, we have computed that

$$\int_{-\infty}^0 \left(f(x-y)g_\varepsilon(y) - f(x+)g(y)\right)dy = \int_{-\infty}^0 g_\varepsilon(y)\left(f(x-y) - f(x+)\right)dy.$$

Remember that $y \leq 0$ where we're integrating. Therefore, $x - y \geq x$. Moreover, by definition

$$\lim_{y\uparrow 0} f(x-y) = f(x+) \implies \lim_{y\uparrow 0} f(x-y) - f(x+) = 0.$$

By definition of limit there exists $y_0 < 0$ such that for all $y \in (y_0, 0)$

$$|f(x-y) - f(x+)| < \widetilde{\delta}.$$

We are using $\widetilde{\delta}$ for now, to indicate that $\widetilde{\delta}$ is going to be something in terms of δ, engineered in such a way that at the end of our argument we get that for ϵ sufficiently small,

$$\left|\int_{-\infty}^0 g_\varepsilon(y)\left(f(x-y) - f(x+)\right)dy\right| < \delta.$$

So, to figure out this $\widetilde{\delta}$, we use our estimate on the part of the integral from y_0 to 0,

$$\left|\int_{y_0}^0 (f(x-y) - f(x+))g_\varepsilon(y)dy\right| \leq \int_{y_0}^0 |f(x-y) - f(x+)||g_\varepsilon(y)|dy$$

$$\leq \widetilde{\delta}\int_{y_0}^0 |g_\varepsilon(y)|dy \leq \widetilde{\delta}\int_{\mathbb{R}} |g_\varepsilon(y)|dy = \widetilde{\delta}||g||.$$

Above, we have used the same substitution trick to see that

$$\int_{\mathbb{R}} |g_\varepsilon(y)|dy = \int_{\mathbb{R}} |g(z)|dz = ||g||,$$

where $||g||$ is the $L^1(\mathbb{R})$ norm of g. By assumption, $g \in L^1(\mathbb{R})$, so this L^1 norm is finite. Moreover, because we know that

$$\int_{\mathbb{R}} g(y)dy = 1,$$

we know that

$$||g|| = \int_{\mathbb{R}} |g(y)|dy \geq \left|\int_{\mathbb{R}} g(y)dy\right| = 1.$$

Hence, I propose setting

$$\widetilde{\delta} = \frac{\delta}{2||g||}.$$

Note that we're not dividing by zero, by the above observation that $||g|| \geq 1$. So, this is a perfectly decent number. Then, we have the estimate (repeating the above estimate)

$$\left| \int_{y_0}^0 (f(x-y) - f(x+))g_\varepsilon(y)dy \right| \leq \int_{y_0}^0 |f(x-y) - f(x+)||g_\varepsilon(y)|dy$$
$$\leq \widetilde{\delta} \int_{y_0}^0 |g_\varepsilon(y)|dy \leq \widetilde{\delta} \int_{\mathbb{R}} |g_\varepsilon(y)|dy = \widetilde{\delta}||g|| = \frac{\delta}{2}.$$

To complete the proof, we just need to estimate the other part of the integral, from $-\infty$ to y_0. It is important to remember that

$$y_0 < 0.$$

So, we wish to estimate

$$\left| \int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\varepsilon(y)dy \right|.$$

Here we need to consider the two possible cases given in the statement of the theorem separately. First, let us assume that f is bounded, which means that there exists $M > 0$ such that $|f(x)| \leq M$ holds for all $x \in \mathbb{R}$. Hence

$$|f(x-y) - f(x+)| \leq |f(x-y)| + |f(x+)| \leq 2M.$$

So, we have the estimate

$$\left| \int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\varepsilon(y)dy \right| \leq \int_{-\infty}^{y_0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy \leq 2M \int_{-\infty}^{y_0} |g_\varepsilon(y)|dy.$$

We shall do a substitution now, letting $z = y/\varepsilon$. Then, as we have computed before,

$$\int_{-\infty}^{y_0} |g_\varepsilon(y)|dy = \int_{-\infty}^{y_0/\varepsilon} |g(z)|dz.$$

Here the limits of integration **do change,** because $y_0 < 0$. Specifically $y_0 \neq 0$, which is why the top limit changes. Now, let's think about what happens as $\varepsilon \to 0$. We're integrating between $-\infty$ and y_0/ε. We know that $y_0 < 0$. So, when we divide it by a really small, but still positive number, like ε, then $y_0/\varepsilon \to -\infty$ as $\varepsilon \to 0$. Moreover, we know that

$$\int_{-\infty}^0 |g(y)|dy < \infty.$$

What this really means is that

$$\lim_{R \to -\infty} \int_R^0 |g(y)|dy = \int_{-\infty}^0 |g(y)|dy < \infty.$$

Hence,

$$\lim_{R \to -\infty} \int_{-\infty}^0 |g(y)|dy - \int_R^0 |g(y)|dy = 0.$$

Of course, we know what happens when we subtract the integral, which shows that

$$\lim_{R \to -\infty} \int_{-\infty}^R |g(y)|dy = 0.$$

Since

$$\lim_{\varepsilon \to 0} y_0/\varepsilon = -\infty,$$

this shows that

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{y_0/\varepsilon}|g(y)|dy = 0.$$

Hence, by definition of limit (see, here it comes again), there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{-\infty}^{y_0/\varepsilon}|g(y)|dy < \frac{\delta}{4(M+1)}.$$

Then, combining this with our estimates, above, which we repeat here,

$$\left|\int_{-\infty}^{y_0}(f(x-y)-f(x+))g_\varepsilon(y)dy\right| \le \int_{-\infty}^{y_0}|f(x-y)-f(x+)||g_\varepsilon(y)|dy \le 2M\int_{-\infty}^{y_0}|g_\varepsilon(y)|dy$$

$$< 2M\frac{\delta}{4(M+1)} < \frac{\delta}{2}.$$

Therefore, we have the estimate that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\left|\int_{-\infty}^{0} g_\varepsilon(y)\left(f(x-y)-f(x+)\right)dy\right|$$

$$\le \int_{-\infty}^{0}|g_\varepsilon(y)||f(x-y)-f(x+)|dy \le \int_{-\infty}^{y_0}|f(x-y)-f(x+)||g_\varepsilon(y)|dy + \int_{y_0}^{0}|f(x-y)-f(x+)||g_\varepsilon(y)|dy$$

$$< \frac{\delta}{2}+\frac{\delta}{2} = \delta.$$

Finally, we consider the other case in the theorem, which is that g vanishes outside a bounded interval. We retain the first part of our estimate, that is

$$\int_{y_0}^{0}|f(x-y)-f(x+)||g_\varepsilon(y)|dy < \frac{\delta}{2}.$$

Next, we again observe that

$$\lim_{\varepsilon\downarrow 0}\frac{y_0}{\varepsilon} = -\infty.$$

By assumption, we know that there exists some $R > 0$ such that

$$g(x) = 0 \forall x \in \mathbb{R} \text{ with } |x| > R.$$

Hence, we may choose ε sufficient small so that

$$\frac{y_0}{\varepsilon} < -R.$$

Specifically, let

$$\varepsilon_0 = \frac{1}{-Ry_0} > 0.$$

Then for all $\varepsilon \in (0, \varepsilon_0)$ we compute that

$$\frac{y_0}{\varepsilon} < -R.$$

Hence for all $y \in (-\infty, y_0/\varepsilon)$ we have $g(y) = 0$. Thus, we compute as before using the substitution $z = y/\varepsilon$,

$$\int_{-\infty}^{y_0}|f(x-y)-f(x+)||g_\varepsilon(y)|dy = \int_{-\infty}^{y_0/\varepsilon}|f(x-\varepsilon z)-f(x+)||g(z)|dz = 0,$$

because $g(z) = 0 \forall z \in (-\infty, y_0/\varepsilon)$. Thus, we have the total estimate that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\left| \int_{-\infty}^{0} g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy \right|$$

$$\leq \int_{-\infty}^{0} |g_\varepsilon(y)||f(x-y)-f(x+)|dy \leq \int_{-\infty}^{y_0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy + \int_{y_0}^{0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy$$

$$< 0 + \frac{\delta}{2} \leq \delta.$$

□

References

[1] R. Mazzeo & J. Rowlett, *A heat trace anomaly on polygons,* Mathematical Proceedings of the Cambridge Philosophical Society, vol. 159 no. 02 (2015) 303–319.
[2] R. Melrose, *The Atiyah-Patodi-Singer Index Theorem,* Research Notes in Mathematics 4. A K Peters, Ltd., (1993).
[3] S. Rosenberg, *The Laplacian on a Riemannian Manifold,* London Math Soc. Student Texts 31, Cambridge University Press, (1997).