HEAT AND GEOMETRY

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I know of two ways to build the heat kernel in a variety of geometric settings which include: closed manifolds, manifolds with smooth boundary, manifolds with cylindrical ends, asymptotically conic (scattering) manifolds, manifolds with edges & conical singularities, asymptotically hyperbolic manifolds (includes hyperbolic manifolds), and maybe am forgetting some other settings where this works. The basic idea is the same for both methods. In the lecture, we shall construct the heat kernel on a closed Riemannian manifold, (M, g). The two ways I know how to do this are:

(1) Construct the heat space which is

$$M_h^2 = [M \times M \times [0, \infty); \operatorname{diag} \times \{t = 0\}, dt].$$

Then construct "the heat operator calculus" which is a graded algebra of pseudodifferential operators. Build up the heat kernel as a certain element of this calculus. The details are contained in [2, Chapter 7]. It is rather technical. The advantage is that this construction works, with certain modifications (basically just blowing up more stuff) in all these geometric settings: manifolds with smooth boundary, manifolds with cylindrical ends, asymptotically conic (scattering) manifolds, manifolds with edges & conical singularities, asymptotically hyperbolic manifolds (includes hyperbolic manifolds), and likely more. The method is rather technical and too long for me to do in the confines of this winter school. However, a lot of the construction is really similar in spirit to the second method:

(2) Build up the heat kernel locally, on little patches of your manifold. Start with a manifold-version of the Euclidean heat kernel, and then solve away the error iteratively, by solving certain PDEs (known as transport equations) and also using Duhamel's principle.

Since the second method is feasible to do here, we'll do it and see what geometric information we can extract. If you decide to undertake the first method, just keep in mind that the "spirit" is very much the same as what we're doing here, you'll just need to get used to some additional abstract mumbo jumbo (the blowups, pseudodifferential operator calculus, half densities, etc). Really, it just requires time to get used to these objects, and then in the end the construction feels a lot like what we will do here. The method we shall use here consists of the following steps:

(1) Start with an approximate heat kernel, known as a *parametrix.* We will use the exponential map and the Euclidean heat kernel to make our parametrix.
(2) It won't really solve the heat equation, but will be a "good approximate solution" in a specific sense (to be made precise in the method).

(3) There's some error between the approximate heat kernel and the real deal, so set up some equations to solve it away. With any luck, the equations will be straightforward to solve in an iterative fashion.
(4) We can therefore iterate our method to get a solution which for small time is really super close to the real deal. We will use Duhamel's principle to do this.
(5) There is some"residual error" which is super small but still needs to get cleaned up. So clean it up.
(6) In the end, we will have obtained the actual heat kernel. We shall prove that it is unique. By our constructive method, we can write down the approximate heat kernels at each step. Thus, with any luck, we can use these approximate heat kernels to compute the real heat kernel *for small time.*

References

[1] R. Mazzeo & J. Rowlett, *A heat trace anomaly on polygons,* Mathematical Proceedings of the Cambridge Philosophical Society, vol. 159 no. 02 (2015) 303–319.
[2] R. Melrose, *The Atiyah-Patodi-Singer Index Theorem,* Research Notes in Mathematics 4. A K Peters, Ltd., (1993).
[3] S. Rosenberg, *The Laplacian on a Riemannian Manifold,* London Math Soc. Student Texts 31, Cambridge University Press, (1997).