HEAT AND GEOMETRY

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1. Pre-Lecture Notes 2018.07.11

Before the very first lecture in this series, you were given the exercise to look up a paper in the very first Journal of Differential Geometry. It is called "curvature and the eigenvalues of the Laplacian." This summer school is about *curvature.* So, maybe it seems weird to basically have a course on the heat kernel as a part of it? If you did the exercise, then you hopefully saw that the heat kernel is the *key* which connects curvature to the eigenvalues of the Laplacian. So, the study of the heat kernel is indeed relevant to our understanding of curvature. We shall get into this today.

(1) We will show that for each $t > 0$, the heat kernel on a compact Riemannian manifold maps $\mathcal{L}^2$ functions to smooth functions.
(2) Next, we will use this to define a compact operator through the heat kernel acting on $\mathcal{L}^2(M)$, for our Riemannian manifold, M.
(3) We will show that this operator is self-adjoint, and therefore we can apply the spectral theorem for compact, self adjoint operators acting on Hilbert spaces (in our case, the Hilbert space, $\mathcal{L}^2(M)$.).
(4) We will use this to prove the spectral theorem for the unbounded operator, Δ.
(5) In this way, we shall obtain an orthonormal basis for $\mathcal{L}^2(M)$ consisting of eigenfunctions of the Laplacian.
(6) We shall give an expression for our heat kernel in terms of this basis.
(7) We will use our construction of the heat kernel from yesterday to show that we can take the trace of the heat kernel. Moreover, we will show that the heat trace admits an asymptotic expansion in powers of t, starting with $t^{-n/2}$, and continuing with higher powers of t, as $t \downarrow 0$.
(8) We shall use the first term in this expansion together with the remainder estimate to obtain Weyl's law for the rate at which the eigenvalues of the Laplacian tend to infinity. This is an alternative proof to Weyl's original proof which was through variational methods (please feel free to ask about that if you're curious :-)
(9) We shall see that the higher terms in the asymptotic expansion of the heat trace as $t \downarrow 0$ are given by integrals of universal polynomials *in the curvature tensor of the Riemannian metric as well as its derivatives!!!* So, indeed curvature is intimately related to the heat kernel!

References

[1] R. Mazzeo & J. Rowlett, *A heat trace anomaly on polygons,* Mathematical Proceedings of the Cambridge Philosophical Society, vol. 159 no. 02 (2015) 303–319.

[2] R. Melrose, *The Atiyah-Patodi-Singer Index Theorem,* Research Notes in Mathematics 4. A K Peters, Ltd., (1993).

[3] S. Rosenberg, *The Laplacian on a Riemannian Manifold,* London Math Soc. Student Texts 31, Cambridge University Press, (1997).