HEAT AND GEOMETRY

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1. Pre-Lecture Notes 2018.07.12

Have you heard the question: *can one hear the shape of a drum?* Do you know the answer? This is the title of an article published by M. Kac in 1966 or 1965, please check the year! In today's lecture we will explore this question and investigate further connections between heat and geometry. Of particular importance is a *locality principle* for the heat kernel. Kac didn't really prove this, but rather stated that for any bounded domain $M \subset \mathbb{R}^n$, the heat kernel on M, denoted by $H(t, x, y)$ satisfies

$$H(t,x,y) = (4\pi t)^{-n/2} e^{-|x-y|^2/(4t)} + \mathcal{O}(t^\infty) \text{t} \downarrow 0,$$

$$\forall x, y \in M \text{ such that } \min\{d(x, \partial M), d(y, \partial M) > \varepsilon,$$

for some fixed constant $\varepsilon > 0$. What does this really mean? Well, it means that the mysterious, unknown heat kernel on M is very close to the Euclidean heat kernel, as long as we stay away from the boundary. For a proof of this fact, see [4]. It is rather amazing because it holds no matter what boundary condition is taken on M as it is a self-adjoint boundary condition.

So, already we see that today's topic is going to be a bit different, because domains in $\mathbb{R}^n$ have *boundary,* whereas our closed and compact Riemannian manifold did not have boundary. In this context it is a bit easier to construct our heat kernel using the eigenvalues of the Laplacian, or at least, that seems the most convenient route to me at this moment. So, let us assume that the Laplacian Δ is acting on some Hilbert space in H^2, for example the Dirichlet boundary condition corresponds to Δ acting on $H^1_0 \cap H^2$, where H^1_0 is the closure of the set of smooth functions whose support is compactly contained in side M, closed with respect to the H^1 norm. For the Neumann boundary condition the Laplacian acts on H^2 (it's not really imposing any boundary condition. It's an interesting fact that not imposing boundary conditions, the eigenfunctions will just on their own end up satisfying the Neumann condition, which says that the normal derivative of the eigenfunction vanishes at the boundary). So, we shall appeal to functional analysis to guarantee that there is a discrete set of eigenvalues

$$\{\lambda_k\}_{k \geq 0}, 0 \leq \lambda_0 \leq \lambda_1 \leq \ldots \uparrow \infty,$$

with eigenfunctions $\{\phi_k\}_{k \geq 0}$ so that

$$\Delta \phi_k = \lambda_k \phi_k \quad \phi_k \in \text{dom}(\Delta).$$

The last condition means that the eigenfunctions satisfy the boundary condition. Then our heat kernel is defined via:

$$H(t,x,y) = \sum_{k \geq 0} e^{-\lambda_k t} \phi_k(x) \phi_k(y).$$

Exercise 1. *Verify that as long as we may integrate by parts (which we may if we have self adjoint boundary conditions on* Δ*), then the same proof of the uniqueness of heat kernels from last lecture is true in this context. Verify that the expression given above is therefore the unique heat kernel on* M*.*

So, we have an expression for the heat kernel, but it is pretty abstract. Indeed, there is no systematic way to compute the eigenfunctions and eigenvalues for a general domain in $\mathbb{R}^n$. There are two ways to surmount this difficulty:

(1) Do the microlocal construction of the heat kernel. This requires blowing up the space $[M \times M \times [0, \infty)]$. This also requires building a parametrix on that space, which shall require a certain "triple space" and also involves defining a heat calculus. It is similar to the construction in [7, Chapter 7].
(2) Use a locality principle and local models to construct the heat kernel, similar to our construction yesterday.

We shall take the second route, because I would need an entire semester-long course to properly do the first method. (Trust me, I've taught this). So, for this we require a locality principle. In [8] we prove a locality principle for the Neumann and Robin boundary conditions. For smoothly bounded domains, the locality principle for Dirichlet and Neumann boundary conditions follows from Lück & Schick [5, Theorem 2.26]. If the domain is in the plane and has corners, then the locality principle essentially follows from [10]. I've not told you exactly what the locality principle means, so if you are curious, then recommended reading is to start with [8] for an explanation of what is meant by locality principle and examples of its applications. For further technical details, the references [5], [10], and [4] are recommended.

In the lecture we will start from this point and proceed to:

(1) Introduce Kac's paper, what's this "hearing the shape of a drum" all about?
(2) Kac's "principle of not feeling the boundary" is an example of a locality principle. We shall recall locality principles as mentioned briefly above in greater detail.
(3) We shall apply the locality principles together with explicit model calculations to compute the short time asymptotic expansion of the heat trace on a polygonal domain. Kac did this as well, but we shall do it using other methods.
(4) Show how Kac used the results for polygons to pass on to smooth domains. There is a "hole" in Kac's proof that the number of holes in a smoothly bounded domain is a spectral invariant. We shall discuss this hole and how to fill it.
(5) Finally we will show that no convex polygonal domain can be isospectral to a smoothly bounded domain.

References

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