HEAT AND GEOMETRY

JULIE ROWLETT

1. Pre-Lecture Notes 2018.07.13

It's Friday the 13th! If you ever hear a fire alarm or have reason to suspect fire go swiftly (and take others with you) to safety! No time to spare! There are two mathematically proven facts about heat:

(1) The speed of propagation of the heat equation is infinite.
(2) The physical underlying process of heat is the Brownian (random) motion of particles.

So, in a nutshell, heat travels super fast and is super random. Yikes! In today's lecture, we shall prove these facts. First we will prove infinite speed of propagation for the heat equation on $\mathbb{R}^n$. To obtain the result for domains in $\mathbb{R}^n$ we shall require a version of the maximum (and minimum) principle for solutions of the heat equation. So, we'll prove that. Finally, we shall require a slight modification of Theorem 1 of [vdbg 19].

Exercise 1. *Work through the proof of* [vdbg 19, Theorem 1] *carefully adapting to the* analyst's *heat equation. This corresponds to changing things by a factor of* 2*, because the probabilist's heat equation uses* $\pm\frac{1}{2}\Delta$ *rather than our* Δ*. Mind the gaps! There is in particular one step, which albeit correct, requires several calculations to verify this step. Can you find this rather giant step? Can you fill in the details?*

We shall use the exercise together with the maximum principle and infinite propagation speed for the heat equation on $\mathbb{R}^n$ to prove infinite speed of propagation for the heat equation on domains in $\mathbb{R}^n$. Then we will move from the macroscopic scale to the microscopic scale...

The notion of heat being the result of the random motion of particles goes back to Einstein (and perhaps his contemporaries or some older guys, but I am not such a history expert). Einstein wrote a series of papers which contain a somewhat hand-wavy proof that the random motion of particles is underlying the flow of heat in particular, and diffusion in general. Interestingly, he also explained how one could use the calculation of the diffusion coefficient based on the laws of physics and thermodynamics to determine the size of molecules.

Exercise 2. *Read* [einstein 1].

We will show how a discrete random walk which tends to a continuous Brownian motion gives rise to the solution of the heat equation. This is essentially the Central Limit Theorem. In this way we will see how the random motion of particles is indeed the stochastic process underlying the heat equation. Finally, we shall return to the central theme of this course: heat and geometry. We shall discuss recent results concerning geometry and curvature which have been obtained using heat kernel type methods.

References

[1] A. Einstein, *Investigations on the theory of the Brownian movement,* ed. with notes by R. Fürth, translated by A. D. Cowper, New York: Dover, (1956).

[2] A. Erdelyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Tables of integral transforms,* McGraw-Hill, (1954).

[3] L. C. Evans, *Partial Differential Equations,* (Second Edition), AMS, (2010).

[4] N. Watson, *A theory of subtemperatures in several variables,* Proc. London Math. Society 26 (1973) 385–417.

[5] B. Fedosov, *Asymptotic formulas for eigenvalues of the Laplacian in a polyhedron.* Doklady Akad. Nauk SSSR, 157 (1964), 536–538.

[6] W. Fulks, *A mean value theorem for the heat equation,* Proc. Amer. Math. Soc. 17(i) (1966) 6–11.

[7] I. S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series, and products,* eighth edition, Elsevier/Academic Press, Amsterdam, (2015).

[8] S. Hawking, *Zeta function regularization of path integrals in curved spacetime.* Comm. Math. Phys. 55 (1977), no. 2, 133–148.

[9] L. Li & A. Strohmaier, *Heat kernel estimates for general boundary problems,* J. Spectral Theory, 6, (2016) 903–919, (arxiv:1604.00784v1).

[10] Z. Lu & J. Rowlett, *The sound of symmetry.* Amer. Math. Monthly 122 (2015), no. 9, 815–835.

[11] Z. Lu & J. Rowlett, *One can hear the corners of a drum.* Bull. Lond. Math. Soc. 48 (2016), no. 1, 85–93.

[12] W. Lück and T. Schick, *L2-torsion of hyperbolic manifolds of finite volume,* Geom. Funct. Anal., 9, no. 3, (1999), 518–567.

[13] R. Mazzeo & J. Rowlett, *A heat trace anomaly on polygons,* Mathematical Proceedings of the Cambridge Philosophical Society, vol. 159 no. 02 (2015) 303–319.

[14] R. Melrose, *The Atiyah-Patodi-Singer Index Theorem,* Research Notes in Mathematics 4. A K Peters, Ltd., (1993).

[15] M. Nursultanov, J. Rowlett, & D. Sher, *How to hear the corners of a drum,* Matrix Annals `https://www.matrix-inst.org.au/2017-matrix-annals/`

[16] B. Osgood, R. Phillips, P. Sarnak, *Extremals of determinants of Laplacians.* J. Funct. Anal. 80 (1988), no. 1, 148–211.

[17] B. Osgood, R. Phillips, P. Sarnak, *Compact isospectral sets of surfaces.* J. Funct. Anal. 80 (1988), no. 1, 212–234.

[18] S. Rosenberg, *The Laplacian on a Riemannian Manifold,* London Math Soc. Student Texts 31, Cambridge University Press, (1997).

[19] M. van den Berg, *Bounds on Green's functions of second-order differential equations,* Journal of Mathematical Physics 22, (1981) 2452–2455.

[20] M. van den Berg & S. Srisatkunarajah, *Heat equation for a region in R2 with polygonal boundary,* J. London Math. Soc. (2) 2 no. 1 (1988) 119–127.

[21] K. Watanabe, *Plane domains which are spectrally determined,* Ann. Global Anal. Geom. 18 , no. 5, (2000), 447–475.

[22] K. Watanabe, *Plane domains which are spectrally determined. II.* J. Inequal. Appl. 7, no. 1, (2002), 25–47.

[23] G. N. Watson, *A treatise on the theory of Bessel functions, Cambridge Mathematical Library,* Cambridge University Press, Cambridge, 1995, Reprint of the second (1944) edition.