Higher Willmore invariants and energies – via boundary calculus

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background:

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Plan:

I. The Willmore energy and invariant.

II. The theory from earlier lectures lead to a boundary calculus for the conformal infinity.

III. This boundary calculus applied to discover and construct new hypersurface invariants and energies that generalise to higher dimensions the Willmore invariant and Willmore energy.
A conformal energy

It was observed by Blaschke 1929, Willmore 1965 and others that

\[ \int_{\Sigma=\text{surf}} H^2 \, dA =: \text{Willmore energy} \quad (\text{or bending energy } 1920) \]

is invariant under the **conformal motions** of \( \mathbb{E}^3 \) (basically \( SO(4,1) \) action). These are the local maps \( \rho : \mathbb{E}^3 \to \mathbb{E}^3 \) that preserve angles.

This property is linked to applications e.g. **solid mechanics** including **cell membranes**, **relativity** and **string theory**, and **geometric analysis** – **Willmore conjecture** concerns absolute minimisers.
The Willmore equation

So the Euler-Lagrange equation (wrt variation of embedding) for the Willmore energy is important:

$$\Delta \Sigma H + 2H(H^2 - K) = 0; \quad K = \text{Gauss curvature.}$$

$\mathcal{B} = \text{Willmore Invariant}$

The WI is an extremely interesting invariant for embeddings into Euclidean space $\mathbb{E}^3$,

$$\iota : \Sigma^2 \longrightarrow \mathbb{E}^3,$$

It is conformally invariant and has linear leading term $\Delta \Sigma H$. A rare quality! One can show that it is a fundamental conformal curvature quantity.

**Question:** What is the meaning of the Willmore invariant? Are there analogues in higher dimensions? This turns out to be linked to the problem of finding all conformal hypersurface invariants!
By embedding a conformal manifold $(\Sigma, \bar{\mathfrak{c}})$ as the boundary at infinity of a Poincaré-Einstein manifold $(\mathcal{M}, g_+)$, the Fefferman-Graham programme initiated in the 1980s led to powerful tools: New approaches to the construction of conformal invariants (of $(\Sigma, \bar{\mathfrak{c}})$), The GJMS operators and $Q$-curvature, scattering theory, ···, Applications to AdS/CFT conjecture, ···

From the early discussion we will see that there is an analogous program in for conformally embedded hypersurfaces. In part this generalises the FG program but in other ways it is a different program.

We work on $\mathcal{M}^{d=n+1}$. Recall that on a conformal manifold $(\mathcal{M}, \mathfrak{c})$ a choice of (generalised) scale $\sigma \in \Gamma(\mathcal{E}[1])$ is equivalent to the scale tractor $I^A$ where $I^A = (\sigma, \nabla^a \sigma, -\frac{1}{d}(\Delta \sigma + J\sigma))$. 
Differential operators by prolonged coupling

On an almost pseudo-Riemannian manifold \((M, c, I)\) there is a canonical differential operator by coupling \(I^A\) to \(D_A\), namely

\[ I \cdot D := I^A D_A. \]

This acts on any weighted tractor bundle, preserving its tensor type but lowering the weight:

\[ I \cdot D : \mathcal{E}^\Phi[w] \to \mathcal{E}^\Phi[w - 1]. \]

It will be useful to define the weight operator \(w\): if \(\beta \in \Gamma(B[w_0])\) we have

\[ w \beta = w_0 \beta. \]

Then on \(\mathcal{E}^\Phi[w]\) we have

\[ I \cdot D \xrightarrow{g^*} \left( -\frac{1}{d}(\Delta \sigma + J \sigma) \quad \nabla^a \sigma \quad \sigma \right) \begin{pmatrix} w(d + 2w - 2) \\ \nabla_a(d + 2w - 2) \\ -(\Delta + J w) \end{pmatrix}. \]

\[ = -\sigma \Delta + (d + 2w - 2)[(\nabla^a \sigma)\nabla_a - \frac{w}{d}(\Delta \sigma)] - \frac{2w}{d}(d + w - 1)\sigma J. \]
The canonical degenerate Laplacian

Now on $\mathcal{M} \setminus \mathcal{Z}(\sigma)$ in the metric $g_{\pm} = \sigma^{-2}g$, with densities trivialised accordingly, we have

$$I \cdot D g_{\pm} \equiv \pm \left( \Delta g_{\pm} + \frac{2w(d + w - 1)}{d} J g_{\pm} \right).$$

In particular if $g_{\pm}$ satisfies $J^{g_{\pm}} = \pm \frac{d}{2}$ (i.e. $Sc^{g_{\pm}} = \pm d(d - 1)$ or equivalently $l^2 = \pm 1$) then, relabeling $d + w - 1 =: s$ and $d - 1 =: n$, we have

$$I \cdot D g_{\pm} \equiv \pm \left( \Delta g_{\pm} \pm s(n - s) \right).$$

so solutions are eigenvectors of the Laplacian (and $s$ is called the spectral parameter) as in scattering theory.

But on $\Sigma = \mathcal{Z}(\sigma) \neq \emptyset$, the conformal infinity, $I \cdot D$ degenerates and there the operator is first order. In particular if the structure is asymptotically ASC in the sense that $l^2 = \pm 1 + \sigma f$, for some smooth $f$, then along $\Sigma$

$$I \cdot D = (d + 2w - 2) \delta_1, \quad \delta_1 \equiv n^a \nabla^a g - w H g = \text{conformal Robin}.$$  

Thus $I \cdot D$ is a degenerate Laplacian, natural to $(\mathcal{M}, c, l)$.  

$(M, c, \sigma)$ be a conformal structure of dimension $d \geq 3$, $\sigma \in \Gamma(\mathcal{E}[1])$ and $I_A = \frac{1}{d} D_A \sigma$ (as usual). Then a direct computation gives

**Lemma**

*Acting on any section of a weighted tractor bundle we have*

$$[I \cdot D, \sigma] = I^2(d + 2w),$$

*where $w$ is the weight operator.*

Thus with *only the restriction that generalised scalar curvature is non-vanishing* we have:

**Proposition (G.-Waldron)**

*Suppose that $(M, c, \sigma)$ is such that $I^2$ is nowhere vanishing. Setting $x := \sigma$, $y := -\frac{1}{I^2} I \cdot D$, and $h := d + 2w$ we obtain the commutation relations*

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h,$$

*of standard $\mathfrak{sl}(2)$-algebra generators.*
### Application: Conformal Laplacian powers

#### Theorem

Let $\mathcal{E}^\Phi$ be any tractor bundle and $k \in \mathbb{Z}_{\geq 1}$. Then, for each $k \in \mathbb{Z}_{\geq 1}$, along $\Sigma = \mathcal{Z}(\sigma)$

$$P_k : \mathcal{E}^\Phi[k - \frac{n}{2}] \to \mathcal{E}^\Phi[-k - \frac{n}{2}]$$
given by

$$P_k := \left(-\frac{1}{l^2} lD\right)^k$$

(1)

is a tangential differential operator, and so determines a canonical differential operator $P_k : \mathcal{E}^\Phi[k - \frac{n}{2}]|_\Sigma \to \mathcal{E}^\Phi[-k - \frac{n}{2}]|_\Sigma$. For $k$ even this takes the form

$$P_k = \Delta^k + \text{lower order terms.}$$

(2)

#### Proof.

From the $\mathfrak{sl}(2)$-identities we have $[x, y^k] = y^{k-1}k(h - k + 1)$. Thus on $\mathcal{E}^\Phi[k - \frac{n}{2}]$

$$P_k(f + \sigma h) = y^k(f + xh) = P_k f + \sigma \tilde{P}_k h.$$ 

So $P_k$ is **tangential**. Expanding the $l$-$D$s yields (2).
Natural boundary problems

Suppose on a conformally compact manifold $M_+$ (with $M_+ \cup \partial M_+ = \overline{M}$) we wish to study solutions to

$$Pf : = \left( \Delta g_+ + \frac{2w(d+w-1)}{d} Jg_+ \right) f = 0.$$ 

E.g. this is what is studied in the usual Poincaré-Einstein scattering program.

Then one needs to fix suitable boundary conditions. E.g. in the case of Riemannian signature one wants some elliptic boundary problem. Since the boundary $\partial M_+$ is at infinity, with $g_+$ singular along $\partial M_+$, this is non-trivial.

But if we view $f$ as the trivialisation of a density of weight $w$ then $Pf \overset{\mathcal{G}}{=} I \cdot Df$ and $I \cdot D$ is well defined on all of $\overline{M}$ (and its smooth extension to $M$ beyond $\partial M_+$). Thus it is natural to study the $I \cdot D$ problem. We do this formally.

First we treat an obvious Dirichlet-like problem where we view $f|_{\Sigma}$ as the initial data.
Asymptotic solutions of the first kind

**Problem**

Given $f|_{\Sigma}$, and an arbitrary extension $f_0$ of this to $\mathcal{E}^\Phi[w_0]$ over $M$, find $f_i \in \mathcal{E}^\Phi[w_0-i]$ (over $M$), $i = 1, 2, \ldots$, so that

$$f^{(\ell)} := f_0 + \sigma f_1 + \sigma^2 f_2 + \cdots + O(\sigma^{\ell+1})$$

solves $l \cdot Df = O(\sigma^\ell)$, off $\Sigma$, for $\ell \in \mathbb{N} \cup \infty$ as high as possible.

$l \cdot Df = 0 \iff -\frac{1}{l^2} l \cdot Df = 0$ so we recast this via $\mathfrak{sl}(2) = \langle x, y, h \rangle$.

**Set** $h_0 = d + 2w_0$. By the identity $[x^k, y] = x^{k-1}k(h + k - 1)$:

$$yf^{(\ell+1)} = yf^{(\ell)} - x^\ell(\ell + 1)(h + \ell)f_{\ell+1} + O(x^{\ell+1}).$$

Now $hf_{\ell+1} = (h_0 - 2(\ell + 1))f_{\ell+1}$, thus

$$yf^{(\ell+1)} = yf^{(\ell)} - x^\ell(\ell + 1)(h_0 - \ell - 2)f_{\ell+1} + O(x^{\ell+1}). \quad (3)$$

By assumption $yf^{(\ell)} = O(x^\ell)$, thus if $\ell \neq h_0 - 2$ we can solve

$$yf^{(\ell+1)} = O(x^{\ell+1})$$

and this uniquely determines $f_{\ell+1}|_{\Sigma}$. 

The obstruction on conformally compact manifolds

So we can solve to all orders provided we do not hit \( \ell = h_0 - 2 \) i.e. provided \( w_0 \notin \{ \frac{k-n}{2} : k \in \mathbb{Z}_{\geq 1} \} \). Otherwise (3) shows that \( \ell = h_0 - 2 \implies yf^{(\ell)} = y(f^{(\ell)} + x^{\ell+1}f_{\ell+1}), \mod O(x^{\ell+1}), \) regardless of \( f_{\ell+1} \). It follows that the map \( f_0 \mapsto x^{-\ell}yf^{(\ell)} \) is tangential and \( x^{-\ell}yf^{(\ell)}|_{\Sigma} \) is the obstruction to solving \( yf^{(\ell+1)} = O(x^{\ell+1}) \). Then by a simple induction this is seen to be a non-zero multiple of \( y^{\ell+1}f_0|_{\Sigma} \):

**Proposition**

If \( \ell = h_0 - 2 \) then the smooth extension is (in general) obstructed by \( P_{\ell+1}f_0|_{\Sigma} \), where \( P_{\ell+1} = (-\frac{1}{i^2} l.Df)^{\ell+1} \) is a tangential operator on densities of weight \( w_0 \).

If \( \ell = h_0 - 2 \) then the extension can be continued with log terms. If \( \overline{M} \) is almost Einstein to sufficiently high order then:

- the odd order \( P_{\ell+1} \) vanish identically; and
- the even order \( P_{\ell+1} \) are the GJMS operators on \( (\partial M_+, \bar{\mathbf{c}}) \).
(Formal) solutions of the second kind

Now we consider the more general type of solution:

**Problem**

Given \( \overline{f}_0|_\Sigma \in \Gamma \mathcal{E}^\Phi [w_0 - \alpha]|_\Sigma \) and an arbitrary extension \( \overline{f}_0 \) of this to \( \Gamma \mathcal{E}^\Phi [w_0 - \alpha] \) over \( \overline{M} \), find \( \overline{f}_i \in \mathcal{E}^\Phi [w_0 - \alpha - i] \) (over \( \overline{M} \)), \( i = 1, 2, \cdots \), so that

\[
\overline{f} := \sigma^\alpha (\overline{f}_0 + \sigma \overline{f}_1 + \sigma^2 \overline{f}_2 + \cdots + O(\sigma^{\ell+1}))
\]

solves \( I \cdot D\overline{f} = O(\sigma^{\ell+\alpha}) \), off \( \partial M_+ \), for \( \ell \in \mathbb{N} \cup \infty \) as high as possible.

Now for \( \alpha \) not integral this Problem takes us outside the realm of the universal enveloping algebra \( \mathcal{U}(\mathfrak{g}) \) and its modules. But it is straightforward to show that for any \( \alpha \in \mathbb{R} \):

\[
[x^\alpha, y] = x^{\alpha-1}\alpha(h + \alpha - 1).
\]
It follows immediately from (5) that $I \cdot D\overline{f} = 0$ has:

- no solution if $\alpha \notin \{0, h_0 - 1\}$, where $h\overline{f} = h_0 \overline{f}$; and
- if $\alpha = h_0 - 1$ and $\overline{f} = \sigma^\alpha f$ then

$$I \cdot D\overline{f} = \sigma^\alpha I \cdot Df$$

So $\overline{f}$ is a solution iff $f$ is!

So in this way **second solutions** arise from **first** and vv.

For $w_0 \notin \{k - n \over 2 : k \in \mathbb{Z}_{\geq 1}\}$, and writing $F = f$, $G = \sigma^{-\alpha} \overline{f}$ we can combine these to a general solution

$$F + \sigma^{h_0 - 1} G = F + \sigma^{n + 2w_0} G$$

or, trivialising the densities on $M_+$ using the generalised scale $\sigma$:

$$f = \sigma^{n-s} F + \sigma^s G = \sigma^{-w_0} (F + \sigma^{h_0 - 1} G)$$

where $s := w_0 + n$. Which is the form of solution used in the **scattering theory** (of Graham-Zworski, Mazzeo-Melrose, ···).

The $\mathfrak{sl}(2)$ approach above solves the asymptotics of $F$ and $G$. 

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The Poincaré-Einstein construction is a tool for studying a conformal manifold \((\Sigma, \bar{\mathfrak{c}})\) holographically. That is for obtaining the invariants and invariant operators of \((\Sigma, \bar{\mathfrak{c}})\) in terms (pseudo-)Riemannian objects on the manifold \(M_+\) of 1 greater dimension that has \(\Sigma = \partial M_+\).

Conversely the scattering theory of \((M, g_+)\) can be understood in terms of non-local conformal operators on the boundary \((\Sigma, \bar{\mathfrak{c}})\).

But requiring \(g_+\) to be Einstein (even asymp. near \(\partial M_+\)) is highly restrictive. It means that the conformal manifold with boundary \((\overline{M}, \mathfrak{c})\) has \(\Sigma = \partial M_+\) totally umbilic, Fialkow vanishes, etcetera.

Here we seek to set up the analogous program for \((\overline{M}, \mathfrak{c})\) a general manifold with boundary.

Thus, given \((\overline{M} = M_+ \cup \partial M_+, \mathfrak{c})\) we need a way to determine a distinguished metric \(g_+ \in \mathfrak{c}|_{M_+}\) on \(M_+\) so that \((M_+, g_+)\) is conformally compact.
Recall a **conformal compactification** of a complete Riemannian manifold \((M^{n+1}, g_+)\) is a manifold \(\overline{M}\) with boundary \(\partial M\) s.t.:

- \(\exists \, \overline{g}\) on \(\overline{M}\), with \(g_+ = r^{-2}\overline{g}\), where
- \(r\) a **defining function** for \(\partial M\): \(\partial M = \mathcal{Z}(r) \& \, dr_p \neq 0 \, \forall p \in \partial M\).

\[ \Rightarrow \] canonically a conformal structure on boundary: \((\partial M, [\overline{g}|_{\partial M}])\).

**Question/variant:** Given \(\overline{g}\) (or really \(c = [\overline{g}]\)) can we find a defining function \(r \in C^\infty(\overline{M})\) for \(\Sigma = \partial M\) s.t.

\[ \text{Sc}(r^{-2}\overline{g}) = -n(n+1)? \]

**NB:** This satisfied for Poincaré-Einstein cf. Loewner-Nirenberg, Aviles and McOwen – related interior problems.
The obstruction density of ACF

Can we solve $\text{Sc}(r^{-2}g) = -n(n + 1)$? formally (i.e. power series) along the boundary? **Answer: No** - in general can get:

**Theorem (Andersson, Chruściel, & Friedrich)**

$$\text{Sc}(r^{-2}g) = -n(n + 1) + r^{n+1} B_n.$$  

*Furthermore (they show)*

$$B_2 = \delta \cdot \delta \cdot \hat{L} + \text{lower order}$$

is a conformal invariant of $\Sigma^2 = \partial M$.

**Theorem.** [G. + Waldron] For $n \geq 2$ $B_n$ is a conformal invariant of $\Sigma = \partial M$, and $B_2 = \text{Willmore Invariant} = \bar{\Delta}H + \text{lower order}$!

- For $n$ even the invariant $B_n$ is **higher order analogue** of $B_2 = B$.

**NB.** The existence of such a higher analogue was not previously obvious as the weight and leading order of $B_n$ means standard tractor/ambient metric approaches fail.
Recasting the problem and holography

Recall the constant scalar curvature condition in terms of scale. A conformal manifold has a canonical conformal metric $g \in S^2 \mathcal{T}^* M[2]$. A metric $g_+ \in \mathfrak{c}$ is equivalent to a scale:

$$g_+ = \sigma^{-2} g \iff \sigma \in \Gamma(\mathcal{E}_+[1]).$$

Via the Thomas-D operator $\bar{D} = \frac{1}{n+1} D$ the scale is equivalent to the scale tractor $I_A := \bar{D} A \sigma,$ and

**Lemma**

$$\text{Sc}(g_+) = -n(n+1) \iff I^2 := h(I, I) = 1$$

So we come to a “conformal Eikonal equation” $(\bar{D} A \sigma)(\bar{D}^A \sigma) = 1$, where $\sigma$ a defining density for $\Sigma$. NB:

- If we could solve uniquely then $\Sigma \hookrightarrow (M, \mathfrak{c})$ determines $g \in \mathfrak{c}$. Then invariants of conf. compact $(M, g_+)$ would be invariants of $\Sigma$. 

Thus to solve the singular Yamabe problem formally we come to the following non-linear problem:

**Problem:** For a conformal manifold \((M, c)\) and an embedding \(\iota: \Sigma \rightarrow M\) solve

\[
I_A I^A = (\bar{D}_A \sigma)(\bar{D}^A \sigma) = 1 + O(\sigma^\ell)
\]

for \(\ell\) as high as possible, and \(\sigma\) a \(\Sigma\) defining density.

A key observation is that the linearisation of \(I^A I_A = 1\) is \(I^A D_A \dot{\sigma} = 0\) – the \(I \cdot D\) problem on \(\mathcal{E}[1]\). Thus \(\exists\) hope that the \(\mathfrak{sl}(2)\) generated by \(x := \sigma, y := -\frac{1}{i^2} I^A D_A\) will again be useful.

Recall from the standard \(\mathfrak{sl}(2)\) identities we have

\[
[I \cdot D, \sigma^{k+1}] = I^2 \sigma^k (k + 1)(n + k + 1 + 2w),
\]

and this allows an inductive solution (using also other tractor identities) that mimics the linear case!
Suppose that $\sigma \in \Gamma(\mathcal{E}[1])$ defines $\Sigma = \partial M_+$ in $(\bar{M}, c)$ and

$$l^2_\sigma = 1 + \sigma^k A_k \quad \text{where} \quad A_k \in \Gamma(\mathcal{E}[-k])$$

is smooth on $M$, and $k \geq 1$, then

- if $k \neq (n + 1)$ then $\exists f_k \in \Gamma(\mathcal{E}[-k])$ s.t. $\sigma' = \sigma + \sigma^{k+1} f_k$ satisfies $l^2_{\sigma'} = 1 + \sigma^{k+1} A_{k+1}$, where $A_{k+1}$ smooth;
- if $k = (n + 1)$ then: $l^2_{\sigma'} = l^2_\sigma + O(\sigma^{n+2})$.

Proof.

Squaring with the tractor metric, using the $\mathfrak{sl}(2)$, etc

$$(\bar{D}\sigma')^2 = (\bar{D}\sigma + \bar{D}(\sigma^{k+1} f_k))^2$$

$$= l^2_\sigma + \frac{2}{n+1} l_\sigma \cdot D(\sigma^{k+1} f_k) + (\bar{D}(\sigma^{k+1} f_k))^2$$

$$= 1 + \sigma^k A_k + \frac{2\sigma^k}{n+1} (k + 1)(n + 1 - k)f_k + O(\sigma^{k+1}).$$
The distinguished defining density

This applies formally off any hypersurface in a Riemannian conformal manifold \((M,c)\) (and even more generally) so we have:

*Theorem (G.-, Waldron arXiv:1506.02723)*

For \(\Sigma^n\) embedded in \((M^{n+1},c)\) there is a distinguished defining density \(\bar{\sigma}\), unique modulo \(+O(\sigma^{n+2})\), s.t.

\[
I_\bar{\sigma}^2 = 1 + \bar{\sigma}^{n+1} B_{\bar{\sigma}}.
\]

Moreover:

\[
B := B_{\bar{\sigma}}|_{\Sigma} \in \Gamma(E_\Sigma \{-n-1\})
\]

is determined by \((M,c,\Sigma)\) and is a natural conformal invariant.

For \(n\) even \(B = 0\) generalises the Willmore equation in that:

\[
B = \tilde{\Delta}^{\frac{n}{2}} H + \text{lower order terms};
\]

while for \(n\) odd \(B\) has no linear leading term.
All submanifold invariants via holography?

The construction can be used to obtain other submanifold invariants: Our Theorem above shows that:

$$(M, c, \Sigma) \text{ determines } \bar{\sigma} \text{ modulo } + O(\sigma^{n+2}).$$

Suppose that $I$ is any coupled conformal invariant of $(M, c, \bar{\sigma})$ involving only the jet $j^{n+1}\bar{\sigma}$. Then along $\Sigma$

$$I \big|_{\Sigma} \text{ is a conformal invariant of } (M, c, \Sigma).$$

This **holographic** approach fails at order $n+2$ precisely because of the existence of the **obstruction invariant** $B$. This is precisely an analogue of the use Fefferman-Graham’s Poincaré and ambient metric constructions to find conformal invariants – that fails at order $n+1$ because of **Bach** $B_{ab}$ in dimension 4 and the **Fefferman-Graham obstruction tensor** in higher even dimensions.
Recall on any almost Riemannian manifold \((M, c, I)\) we had:

**Theorem**

Let \(\mathcal{E}^\Phi\) be any tractor bundle and \(k \in \mathbb{Z}_{\geq 1}\). Then, for each \(k \in \mathbb{Z}_{\geq 1}\), along \(\Sigma = \mathcal{Z}(\sigma)\)

\[
P^\sigma_k : \mathcal{E}^\Phi[\frac{k - n}{2}] \to \mathcal{E}^\Phi[\frac{-k - n}{2}]
\]

given by

\[
P^\sigma_k := \left(-\frac{1}{l^2}I \cdot D\right)^k
\]

is a tangential differential operator, and so determines a canonical differential operator \(P^\sigma_k : \mathcal{E}^\Phi[\frac{k - n}{2}]|_\Sigma \to \mathcal{E}^\Phi[\frac{-k - n}{2}]|_\Sigma\). For \(k\) even this takes the form

\[
P_k = \Delta^k + \text{lower order terms}.
\]

Because \((M, c, \Sigma)\) determines \(\bar{\sigma}\) modulo \(O(\sigma^{n+2})\), we have:

**Theorem**

For \(k \leq n = d - 1\) the operators \(P_k\) are determined canonically by the data \((M, c, \Sigma)\).
Higher Willmore energies

For suitable regularisations $\overline{M}_\epsilon$ of conformally compact manifolds $\overline{M}$:

$$\text{Vol}_\epsilon = \int_{\overline{M}_\epsilon} \sqrt{g_+} = \frac{v_n}{\epsilon^n} + \cdots + \frac{v_1}{\epsilon} + A \log \epsilon + V_{\text{ren}} + O(\epsilon).$$

**Theorem (Graham 2016: arXiv:1606.00069)**

If $g_+ = \bar{\sigma}^{-2}g$, i.e. it is the approximate solution of the singular Yamabe problem then $A$ a conformal invariant of $\Sigma \hookrightarrow M$ and

$$\frac{\delta A}{\delta \Sigma} = \frac{d(d-2)}{2} \beta_n.$$

So the anomaly term in the renormalised volume expansion provides an energy with functional gradient the obstruction density, in other words an energy generalising the Willmore energy.
Extrinsic $Q$-curvature and the anomaly

In fact – also in analogy with the treatment of Poincaré-Einstein manifolds – there is nice local quantity giving the anomaly:

**Theorem (G.- Waldron arXiv:1603.07367)**

$$A = \frac{1}{(d-1)!(d-2)!} \int_{\Sigma} Q$$

where, with $\tau \in \Gamma E_+[1]$ a scale giving the boundary metric, $Q := (-I \cdot D)^n \log \tau$.

- $Q$ here is an **extrinically coupled** $Q$-curvature meaning e.g.

$$Q^{\hat{g}_{\Sigma}} = e^{-nf} (Q^g + P_n f) \text{ where } \hat{g}_{\Sigma} = e^{2f} g_{\Sigma}$$

and for $n$ even

$$P_n = \Delta_{\Sigma}^\frac{n}{2} + \text{lower order terms}; P_n \text{ FSA, and } P_n1 = 0,$$

is an **extrinically coupled** GJMS type operator. $Q$ and $P_n$ are from G.-, Waldron arXiv:1104.2991 = Indiana U.M.J. 2014.
Idea of proof

Use a Heaviside function $\theta$ to "cut off" an integral over all $\overline{M}$

$$\text{Vol}_\epsilon = \int_{\overline{M}} \frac{dV_{\tau}}{\sigma^d} \theta\left(\frac{\sigma}{\tau} - \epsilon\right).$$

Then the divergent terms and anomaly are given by

$$v_k \sim \frac{d^{d-1-k}}{d\epsilon^{d-1-k}} \left( \epsilon^d \frac{d}{d\epsilon} \text{Vol}_\epsilon \right) \bigg|_{\epsilon=0},$$

So

$$v_k \sim \int_{\overline{M}} \frac{\delta^{d-1-k}(\sigma)}{\tau^k} \quad \text{and} \quad A \sim \int_{\overline{M}} \delta^{d-2}(\sigma) I \cdot D \log \tau$$

Then via identities, and the $sl(2)$ again

$$v_k \sim \int_{\Sigma} (I \cdot D)^{d-k-1} \frac{1}{\tau^k} \quad \text{and} \quad A \sim \int_{\Sigma} (I \cdot D)^{d-1} \log \tau$$
An “Obata type” question/conjecture

The Obata Theorem states (more than):

**Theorem (Obata)**

If $g$ is a metric on the round sphere $S^n$ that is conformal to the standard metric $\bar{g}$ and has constant scalar curvature, then $g$ is Einstein.

NB: The metric here takes the form $g = \sigma^{-2} \bar{g}$ where $\sigma$ is a smooth nowhere vanishing function, and the condition that it has constant scalar curvature is that the scale tractor satisfies $I_\sigma^2 = \text{constant}$ (where recall $I = \bar{D}\sigma$). Thus related to the Obata Theorem there is a very nice more general question:

**Question**

Let $(S^n, \bar{g})$ be the usual round sphere. Let $\sigma \in C^\infty(S^n)$, possibly with $\mathcal{Z}(\sigma)$ non-empty, such that $I_\sigma^2 = \text{constant}$. Then is $g = \sigma^{-2} \bar{g}$ necessarily Einstein on $S^n \setminus \mathcal{Z}(\sigma)$?
Above question is especially interesting if \( I^2 = \text{const.} \neq 0 \) then:

**Theorem (G. 2006)**

\[ \mathcal{Z}(\sigma), \text{ if } \mathcal{Z} \neq \emptyset, \text{ is a smoothly embedded separating hypersurface that gives a conformal compactification of } g = \sigma^{-2} \bar{g}. \]

Thus with the earlier mentioned Theorem of G. + Waldron:

**Corollary**

Let \((S^n, \bar{g})\) be the usual round sphere. Let \( \sigma \in C^\infty(S^n) \), with \( \mathcal{Z}(\sigma) \) non-empty, such that \( I_\sigma^2 = \text{const.} \neq 0 \). Then \( \mathcal{Z}(\sigma) \) is a (higher) Willmore hypersurface.

\( \exists \) solutions to \( I_\sigma^2 = \text{const.} \neq 0 \) with \( \mathcal{Z}(\sigma) \) totally umbilic. Hence:

**Question**

Let \((S^n, \bar{g})\) be the usual round sphere. Let \( \sigma \in C^\infty(S^n) \), possibly with \( \mathcal{Z}(\sigma) \) non-empty, such that \( I_\sigma^2 = \text{constant} \). Then is \( \mathcal{Z}(\sigma) \) necessarily totally umbilic?
Compactification and boundary calc
Rod Gover


Compactification and boundary calc