Higher Willmore invariants and energies – via boundary calculus

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background:
G-., Waldon, **Commun. Contemp. Math., in press, arXiv:1611.08345**, and **Renormalized Volume, CMP, 2017**, and **arXiv:1506.02723**; Curry, **G-.** $\cdots$ Conformal Geometry $\cdots$ GR$\cdots$, LMS Series, Cambridge, arXiv:1412.7559
Čap, **G-.** Hammerl: Holonomy reductions etc, *Duke Math. J.* **163** (2014) 1035–1070.
G-. *J. Geom. Phys.*, **60** (2010), 182–204.

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Lecture 5

Plan:

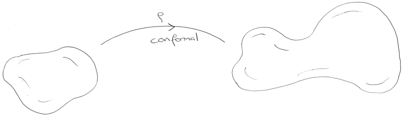
- I. The Willmore energy and invariant.
- II. The theory from earlier lectures lead to a boundary calculus for the conformal infinity.
- III. This boundary calculus applied to discover and construct new hypersurface invariants and energies that generalise to higher dimensions the Willmore invariant and Willmore energy.

A conformal energy

It was observed by Blaschke 1929, Willmore 1965 and others that

$$\int_{\Sigma=surf} H^2 \, dA =: \textbf{Willmore energy} \quad \text{(or bending energy 1920)}$$

is invariant under the **conformal motions** of $\mathbb{E}^3$ (basically $SO(4,1)$ action). These are the local maps $\rho : \mathbb{E}^3 \to \mathbb{E}^3$ that preserve angles.



This property is linked to applications e.g. **solid mechanics** including **cell membranes**, **relativity** and **string theory**, and **geometric analysis** – Willmore conjecture concerns absolute minimisers.

The Willmore equation

So the Euler-Lagrange equation (wrt variation of embedding) for the Willmore energy is important:

$$\underbrace{\Delta_\Sigma H + 2H(H^2 - K)}_{\boxed{\mathcal{B} = \textbf{W}\text{illmore } \textbf{I}\text{nvariant}}} = 0; \qquad K = \text{Gauss curvature.}$$

The WI is an **extremely interesting invariant** for embeddings into Euclidean space $\mathbb{E}^3$,

$$\iota : \Sigma^2 \longrightarrow \mathbb{E}^3,$$

It is **conformally invariant** and has **linear leading term** $\Delta_\Sigma H$. A rare quality! One can show that it is a **fundamental conformal curvature quantity**.

Question: What is the meaning of the Willmore invariant? Are there analogues in higher dimensions? This turns out to be linked to the problem of finding **all** conformal hypersurface **invariants**!

Part II: Boundary calculus

By embedding a conformal manifold $(\Sigma, \bar{\mathbf{c}})$ as the boundary at infinity of a Poincaré-Einstein manifold (M, g_+), the Fefferman-Graham programme initiated in the 1980s led to powerful tools: New approaches to the construction of conformal invariants (of $(\Sigma, \bar{\mathbf{c}})$), The GJMS operators and Q-curvature, scattering theory, $\cdots$, Applications to AdS/CFT conjecture, $\cdots$

From the early discussion we will see that there is an **analogous program** in for conformally embdedded hypersurfaces. In part this **generalises** the FG program but in other ways it is a **different program**.

We work on $M^{d=n+1}$. Recall that on a conformal manifold $(M, \mathbf{c})$ a choice of (generalised) scale $\sigma \in \Gamma(\mathcal{E}[1])$ is equivalent to the **scale tractor** I^A where $I^A = (\sigma,\ \nabla^a \sigma, -\frac{1}{d}(\Delta\sigma + J\sigma))$.

Differential operators by prolonged coupling

On an almost pseudo-Riemannian manifold $(M, \mathbf{c}, I)$ there is a canonical differential operator by **coupling** I^A **to** D_A , namely

$$I\cdot D := I^A D_A.$$

This acts on any weighted tractor bundle, preserving its tensor type but lowering the weight:

$$I\cdot D : \mathcal{E}^\Phi[w] \to \mathcal{E}^\Phi[w-1].$$

It will be useful to define define the *weight operator* $\mathbf{w}$: if $\beta \in \Gamma(\mathcal{B}[w_0])$ we have

$$\mathbf{w}\,\beta = w_0 \beta.$$

Then on $\mathcal{E}^\Phi[w]$ we have

$$I\cdot D \stackrel{g}{=} \begin{pmatrix} -\frac{1}{d}(\Delta\sigma + \mathrm{J}\sigma) & \nabla^a\sigma & \sigma \end{pmatrix} \begin{pmatrix} \mathbf{w}(d+2\mathbf{w}-2) \\ \nabla_a(d+2\mathbf{w}-2) \\ -(\Delta + \mathrm{J}\mathbf{w}) \end{pmatrix}.$$

$$= -\sigma\Delta + (d+2w-2)[(\nabla^a\sigma)\nabla_a - \frac{w}{d}(\Delta\sigma)] - \frac{2w}{d}(d+w-1)\sigma\mathrm{J}$$

The canonical degenerate Laplacian

Now on $M \setminus \mathcal{Z}(\sigma)$ in the metric $g_{\pm} = \sigma^{-2}\boldsymbol{g}$, with densities trivialised accordingly, we have

$$I\cdot D \stackrel{g_{\pm}}{=} \mp\Big(\Delta^{g_{\pm}} + \frac{2w(d+w-1)}{d}\mathrm{J}^{g_{\pm}}\Big).$$

In particular if $g_{\pm}$ satisfies $\mathrm{J}^{g_{\pm}} = \mp\frac{d}{2}$ (*i.e.* $\mathrm{Sc}^{g_{\pm}} = \mp d(d-1)$ or equivalently $I^2 = \pm 1$) then, relabeling $d+w-1 =: s$ and $d-1 =: n$, we have

$$\boxed{I\cdot D \stackrel{g_{\pm}}{=} \mp(\Delta^{g_{\pm}} \pm s(n-s))}.$$

so solutions are **eigenvectors of the Laplacian** (and s is called the **spectral parameter**) as in **scattering theory**.

But on $\Sigma = \mathcal{Z}(\sigma) \neq \emptyset$, the conformal infinity, $I\cdot D$ degenerates and there the operator is first order. In particular if the structure is asymptotically ASC in the sense that $I^2 = \pm 1 + \sigma f$, for some smooth f, then along Σ

$$\boxed{I\cdot D = (d+2w-2)\delta_1}, \qquad \delta_1 \stackrel{g}{=} n^a\nabla_a^g - wH^g = \text{ conformal Robin}$$

Thus $I\cdot D$ is a **degenerate Laplacian**, natural to $(M, \mathbf{c}, I)$.

The $\mathfrak{sl}(2)$-algebra

$(M, \mathbf{c})$ be a conformal structure of dimension $d \geq 3$, $\sigma \in \Gamma(\mathcal{E}[1])$ and $I_A = \frac{1}{d} D_A \sigma$ (as usual). Then a direct computation gives

Lemma

Acting on any section of a weighted tractor bundle we have

$$[I \cdot D, \sigma] = I^2(d + 2\mathbf{w}),$$

where $\mathbf{w}$ *is the weight operator.*

Thus with **only the restriction that generalised scalar curvature is non-vanishing** we have:

Proposition (G.-Waldron)

Suppose that (M, c, σ) *is such that* I^2 *is nowhere vanishing. Setting* $x := \sigma$, $y := -\frac{1}{I^2} I \cdot D$, *and* $h := d + 2\mathbf{w}$ *we obtain the commutation relations*

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h,$$

of standard $\mathfrak{sl}(2)$*-algebra generators.*

Application: Conformal Laplacian powers

Theorem

Let $\mathcal{E}^\Phi$ be any tractor bundle and $k \in \mathbb{Z}_{\geq 1}$. Then, for each $k \in \mathbb{Z}_{\geq 1}$, along $\Sigma = \mathcal{Z}(\sigma)$

$$P_k : \mathcal{E}^\Phi[\frac{k-n}{2}] \to \mathcal{E}^\Phi[\frac{-k-n}{2}] \quad \text{given by} \quad P_k := \left(-\frac{1}{I^2} I\!\!D\right)^k \quad (1)$$

is a tangential differential operator, and so determines a canonical differential operator $P_k : \mathcal{E}^\Phi[\frac{k-n}{2}]|_\Sigma \to \mathcal{E}^\Phi[\frac{-k-n}{2}]|_\Sigma$. For k even this takes the form

$$P_k = \overline{\Delta}^k + \text{lower order terms}. \quad (2)$$

Proof.

From the $\mathfrak{sl}(2)$-identities we have $[x, y^k] = y^{k-1}k(h-k+1)$.
Thus on $\mathcal{E}^\Phi[\frac{k-n}{2}]$

$$P_k(f + \sigma h) = y^k(f + xh) = P_k f + \sigma \widetilde{P}_k h.$$

So P_k is **tangential**. Expanding the $I \cdot D$s yields (2). □

Natural boundary problems

Suppose on a conformally compact manifold M_+ (with $M_+ \cup \partial M_+ = \overline{M}$) we wish to study solutions to

$$Pf := \left(\Delta^{g_+} + \frac{2w(d+w-1)}{d} \mathsf{J}^{g_+}\right) f = 0.$$

E.g. this is what is studied in the usual Poincaré-Einstein scattering program.

Then one needs to fix suitable boundary conditions. E.g. in the case of Riemannian signature one wants some elliptic boundary problem. Since the boundary ∂M_+ is at infinity, with g_+ singular along ∂M_+, this is non-trivial.

But if we view f as the trivialisation of a density of weight w then $Pf \stackrel{g_+}{=} I{\cdot}Df$ and $I{\cdot}D$ is well defined on all of $\overline{M}$ (and its smooth extension to M beyond ∂M_+). Thus it is natural to study the $I{\cdot}D$ problem. We do this **formally**.

First we treat an obvious Dirichlet-like problem where we view $f|_\Sigma$ as the initial data.

Asymptotic solutions of the first kind

Problem

Given $f|_\Sigma$, and an arbitrary extension f_0 of this to $\mathcal{E}^\Phi[w_0]$ over M, find $f_i \in \mathcal{E}^\Phi[w_0 - i]$ (over M), $i = 1, 2, \cdots$, so that

$$f^{(\ell)} := f_0 + \sigma f_1 + \sigma^2 f_2 + \cdots + O(\sigma^{\ell+1})$$

solves $I\cdot Df = O(\sigma^\ell)$, off Σ, for $\ell \in \mathbb{N} \cup \infty$ as high as possible.

$I\cdot Df = 0 \Leftrightarrow -\frac{1}{I^2} I\cdot Df = 0$ so we recast this via $\mathfrak{sl}(2) = \langle x, y, h\rangle$.
Set $h_0 = d + 2w_0$. By the identity $[x^k, y] = x^{k-1}k(h + k - 1)$:

$$yf^{(\ell+1)} = yf^{(\ell)} - x^\ell(\ell+1)(h+\ell)f_{\ell+1} + O(x^{\ell+1}).$$

Now $hf_{\ell+1} = \big(h_0 - 2(\ell+1)\big)f_{\ell+1}$, thus

$$yf^{(\ell+1)} = yf^{(\ell)} - x^\ell(\ell+1)(h_0 - \ell - 2)f_{\ell+1} + O(x^{\ell+1}). \qquad (3)$$

By assumption $yf^{(\ell)} = O(x^\ell)$, thus if $\boxed{\ell \neq h_0 - 2}$ we can solve $yf^{(\ell+1)} = O(x^{\ell+1})$ and this **uniquely determines** $f_{\ell+1}|_\Sigma$.

The obstruction on conformally compact manifolds

So we can solve to all orders provided we do not hit $\ell = h_0 - 2$ i.e. provided $w_0 \notin \{\frac{k-n}{2} \ : \ k \in \mathbb{Z}_{\geq 1}\}$. Otherwise (3) shows that

$$\ell = h_0 - 2 \quad \Rightarrow \quad yf^{(\ell)} = y\big(f^{(\ell)} + x^{\ell+1} f_{\ell+1}\big), \qquad \text{modulo } O(x^{\ell+1}),$$

regardless of $f_{\ell+1}$. It follows that the map $f_0 \mapsto x^{-\ell} y f^{(\ell)}$ is tangential and $x^{-\ell} y f^{(\ell)}|_\Sigma$ is the obstruction to solving $yf^{(\ell+1)} = O(x^{\ell+1})$. Then by a simple induction this is seen to be a non-zero multiple of $y^{\ell+1} f_0|_\Sigma$:

Proposition

If $\ell = h_0 - 2$ then the smooth extension is (in general) obstructed by $P_{\ell+1} f_0|_\Sigma$, where $P_{\ell+1} = (-\frac{1}{I^2} I \cdot Df)^{\ell+1}$ is a tangential operator on densities of weight w_0.

If $\ell = h_0 - 2$ then the extension can be continued with **log terms**.
If $\overline{M}$ is **almost Einstein** to sufficiently high order then:

- the **odd order** $P_{\ell+1}$ **vanish identically**; and
- the **even order** $P_{\ell+1}$ are the **GJMS operators** on $(\partial M_+, \bar{\mathbf{c}})$.

(Formal) solutions of the second kind

Now we consider the more general type of solution:

Problem

Given $\overline{f}_0|_\Sigma \in \Gamma\mathcal{E}^\Phi[w_0 - \alpha]|_\Sigma$ and an arbitrary extension $\overline{f}_0$ of this to $\Gamma\mathcal{E}^\Phi[w_0 - \alpha]$ over $\overline{M}$, find $\overline{f}_i \in \mathcal{E}^\Phi[w_0 - \alpha - i]$ (over $\overline{M}$), $i = 1, 2, \cdots$, so that

$$\overline{f} := \sigma^\alpha(\overline{f}_0 + \sigma\,\overline{f}_1 + \sigma^2\,\overline{f}_2 + \cdots + O(\sigma^{\ell+1})) \qquad (4)$$

solves $I \cdot D\overline{f} = O(\sigma^{\ell+\alpha})$, off ∂M_+, for $\ell \in \mathbb{N} \cup \infty$ as high as possible.

Now for α not integral this Problem takes us outside the realm of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and its modules. But it is straightforward to show that for any $\alpha \in \mathbb{R}$:

$$[x^\alpha, y] = x^{\alpha-1}\alpha(h + \alpha - 1). \qquad (5)$$

It follows immediately from (5) that $I \cdot D\overline{f} = 0$ has:

- no solution if $\alpha \notin \{0, h_0 - 1\}$, where $h\overline{f} = h_0\overline{f}$; and
- if $\alpha = h_0 - 1$ and $\overline{f} = \sigma^\alpha f$ then

$$I{\cdot}D\overline{f} = \sigma^\alpha I{\cdot}Df \qquad \boxed{\text{So } \overline{f} \text{ is a solution iff } f \text{ is!}}$$

So in this way **second solutions** arise from **first** and vv.

For $w_0 \notin \{\frac{k-n}{2} \ : \ k \in \mathbb{Z}_{\geq 1}\}$, and writing $F = f$, $G = \sigma^{-\alpha}\overline{f}$ we can combine these to a general solution

$$F + \sigma^{h_0 - 1}G = F + \sigma^{n+2w_0}G$$

or, trivialising the densities on M_+ using the generalised scale σ:

$$f = \sigma^{n-s}F + \sigma^s G = \sigma^{-w_0}(F + \sigma^{h_0-1}G)$$

where $s := w_0 + n$. Which is the form of solution used in the **scattering theory** (of Graham-Zworski, Mazzeo-Melrose, $\cdots$). The $\mathfrak{sl}(2)$ approach above solves the asymptotics of F and G.

Part III: The Loewner-Nirenberg problem and higher Willmore

The Poincaré-Einstein construction is a tool for studying a conformal manifold $(\Sigma, \bar{\mathbf{c}})$ **holographically**. That is for obtaining the invariants and invariant operators of $(\Sigma, \bar{\mathbf{c}})$ in terms (pseudo-)Riemannian objects on the manifold M_+ of 1 greater dimension that has $\Sigma = \partial M_+$.

Conversely the scattering theory of (M, g_+) can be understood in terms of non-local conformal operators on the boundary $(\Sigma, \bar{\mathbf{c}})$.

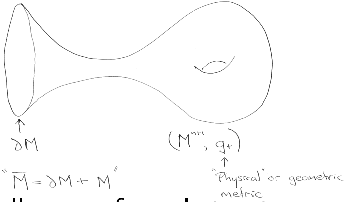
But requiring g_+ to be Einstein (even asymp. near ∂M_+) is **highly restrictive**. It means that the conformal manifold with boundary $(\overline{M}, \mathbf{c})$ has $\Sigma = \partial M_+$ totally umbilic, Fialkow vanishes, etcetera.

Here we seek to set up the analogous program for $(\overline{M}, \mathbf{c})$ **a general manifold with boundary**.

Thus, given $(\overline{M} = M_+ \cup \partial M_+, \mathbf{c})$ we **need a way to determine a distinguished metric** $g_+ \in \mathbf{c}|_{M_+}$ on M_+ so that (M_+, g_+) is conformally compact.

Recall a **conformal compactification** of a complete Riemannian manifold (M^{n+1}, g_+) is a manifold $\overline{M}$ with boundary ∂M s.t.:

- $\exists\ \overline{g}$ on $\overline{M}$, with $g_+ = r^{-2}\overline{g}$, where
- r a **defining function** for ∂M: $\partial M = \mathcal{Z}(r)$ & $dr_p \neq 0\ \forall p \in \partial M$.



$\Rightarrow$ canonically a conformal structure on boundary: $(\partial M, [\overline{g}|_{\partial M}])$.

Question/variant: Given $\overline{g}$ (or really $\mathbf{c} = [\overline{g}]$) can we find a defining function $r \in C^\infty(\overline{M})$ for $\Sigma = \partial M$ s.t.

$$\mathrm{Sc}(r^{-2}\overline{g}) = -n(n+1)?$$ NB: This satisfied for Poincaré-Einstein

cf. Loewner-Nirenberg, Aviles and McOwen – related interior problems.

The obstruction density of ACF

Can we solve $\mathrm{Sc}(r^{-2}\overline{g}) = -n(n+1)$? formally (i.e. power series) along the boundary? **Answer: No** - in general can get:

Theorem (Andersson, Chruściel, & Friedrich)

$$\mathrm{Sc}(r^{-2}\overline{g}) = -n(n+1) + r^{n+1}\mathcal{B}_n.$$

Furthermore (they show)

$$\mathcal{B}_2 = \delta \cdot \delta \cdot \mathring{L} + \textit{lower order}$$

is a conformal invariant of $\Sigma^2 = \partial M$.

Theorem.[G. + Waldron] For $n \geq 2$ $\mathcal{B}_n$ is a conformal invariant of $\Sigma = \partial M$, and $\mathcal{B}_2 =$ **Willmore Invariant** $= \overline{\Delta} H +$ lower order!

•For n even the invariant $\mathcal{B}_n$ is **higher order analogue** of $\mathcal{B}_2 = \mathcal{B}$.

NB. The existence of such a higher analogue was not previously obvious as the weight and leading order of $\mathcal{B}_n$ means standard tractor/ambient metric approaches fail.

Recasting the problem and holography

Recall the constant scalar curvature condition in terms of scale. A conformal manifold has a canonical conformal metric $\boldsymbol{g} \in S^2T^*M[2]$. A metric $g_+ \in \mathbf{c}$ is equivalent to a **scale**:

$$\boxed{g_+ = \sigma^{-2}\boldsymbol{g}} \quad \Leftrightarrow \quad \boxed{\sigma \in \Gamma(\mathcal{E}_+[1]).}$$

Via the Thomas-D operator $\bar{D} = \frac{1}{n+1}D$ the scale is equivalent to the

$$\boxed{\textbf{scale tractor} \quad I_A := \bar{D}_A\sigma,} \qquad \text{and}$$

Lemma

$\mathrm{Sc}(g_+) = -n(n+1) \Leftrightarrow I^2 := h(I, I) = 1$

So we come to a "conformal Eikonal equation" $(\bar{D}_A\sigma)(\bar{D}^A\sigma) = 1$, where σ a **defining density** for Σ. **NB:**

- If $\boxed{\text{we could solve uniquely then } \Sigma \hookrightarrow (M, \mathbf{c}) \textbf{ determines } g \in \mathbf{c}}$.

Then invariants of conf. compact (M, g_+) would be invariants of Σ.

The conformal Eikonal equation

Thus to solve the singular Yamabe problem formally we come to the following **non-linear** problem:

Problem: For a conformal manifold $(M, \mathbf{c})$ and an embedding $\iota : \Sigma \to M$ solve

$$I_A I^A = (\bar{D}_A \sigma)(\bar{D}^A \sigma) = 1 + O(\sigma^\ell)$$

for ℓ as high as possible, and σ a Σ defining density.

A key observation is that the **linearisation** of $I^A I_A = 1$ is $I^A D_A \dot{\sigma} = 0$ – the $I \cdot D$ problem on $\mathcal{E}[1]$. Thus $\exists$ hope that the $\mathfrak{sl}(2)$ generated by $x := \sigma$, $y := -\frac{1}{I^2} I^A D_A$ will again be useful.

Recall from the standard $\mathfrak{sl}(2)$ identities we have

$$[I \cdot D, \sigma^{k+1}] = I^2 \sigma^k (k+1)(n+k+1+2\boldsymbol{w}),$$

and this allows an inductive solution (using also **other tractor identities**) that mimics the linear case!

Lemma

Suppose that $\underline{\sigma \in \Gamma(\mathcal{E}[1]) \textit{ defines } \Sigma = \partial M_+ \textit{ in } (\overline{M}, \mathbf{c})}$ *and*

$$I_\sigma^2 = 1 + \sigma^k A_k \quad \textit{where} \quad A_k \in \Gamma(\mathcal{E}[-k])$$

is smooth on M, *and* $k \geq 1$, *then*

- *if* $k \neq (n+1)$ *then* $\exists\ f_k \in \Gamma(\mathcal{E}[-k])$ *s.t.* $\sigma' := \sigma + \sigma^{k+1} f_k$ *satisfies* $I_{\sigma'}^2 = 1 + \sigma^{k+1} A_{k+1}$, *where* A_{k+1} *smooth;*
- *if* $k = (n+1)$ *then:* $I_{\sigma'}^2 = I_\sigma^2 + O(\sigma^{n+2})$.

Proof.

Squaring with the tractor metric, using the $\mathfrak{sl}(2)$, etc

$$\begin{aligned}(\bar{D}\sigma')^2 &= (\bar{D}\sigma + \bar{D}(\sigma^{k+1} f_k))^2 \\ &= I_\sigma^2 + \frac{2}{n+1} I_\sigma \cdot D(\sigma^{k+1} f_k) + (\bar{D}(\sigma^{k+1} f_k))^2 \\ &= 1 + \sigma^k A_k + \frac{2\sigma^k}{n+1}(k+1)(n+1-k) f_k + O(\sigma^{k+1}).\end{aligned}$$

□

The distinguished defining density

This applies formally off any hypersurface in a Riemannian conformal manifold $(M, \mathbf{c})$ (and even more generally) so we have:

Theorem (G.-, Waldron arXiv:1506.02723)

For Σ^n embedded in $(M^{n+1}, \mathbf{c})$ there is a distinguished defining density $\bar{\sigma}$, **unique** *modulo $+O(\sigma^{n+2})$, s.t.*

$$I_{\bar{\sigma}}^2 = 1 + \bar{\sigma}^{n+1} \mathcal{B}_{\bar{\sigma}}.$$

Moreover:

$$\mathcal{B} := \mathcal{B}_{\bar{\sigma}}|_{\Sigma} \in \Gamma(\mathcal{E}_{\Sigma}[-n-1])$$

is determined by $(M, \mathbf{c}, \Sigma)$ and is a **natural conformal invariant**.

For n even $\mathcal{B} = 0$ generalises the Willmore equation in that:

$$\mathcal{B} = \bar{\Delta}^{\frac{n}{2}} H + \text{lower order terms};$$

while for n odd $\mathcal{B}$ has no linear leading term.

All submanifold invariants via holography?

The construction can be used to obtain other submanifold invariants: Our Theorem above shows that:

$$(M, \mathbf{c}, \Sigma) \quad \text{determines} \quad \bar{\sigma} \quad \text{modulo} + O(\sigma^{n+2}).$$

Suppose that $\mathcal{I}$ is any coupled conformal invariant of $(M, \mathbf{c}, \bar{\sigma})$ involving only the jet $j^{n+1}\bar{\sigma}$. Then along Σ

$$\boxed{\mathcal{I}\big|_{\Sigma} \quad \text{is a conformal invariant of} \quad (M, \mathbf{c}, \Sigma).}$$

This **holographic** approach fails at order $n+2$ precisely because of the existence of the **obstruction invariant** $\mathcal{B}$. This is precisely an analogue of the use Fefferman-Graham's Poincaré and ambient metric constructions to find conformal invariants – that fails at order $n+1$ because of **Bach** B_{ab} in dimension 4 and the **Fefferman-Graham obstruction tensor** in higher even dimensions.

Extrinsically coupled GJMS operators

Recall on any almost Riemannian manifold (M, c, I) we had:

Theorem

Let $\mathcal{E}^\Phi$ be any tractor bundle and $k \in \mathbb{Z}_{\geq 1}$. Then, for each $k \in \mathbb{Z}_{\geq 1}$, along $\Sigma = \mathcal{Z}(\sigma)$

$$P_k^\sigma : \mathcal{E}^\Phi[\frac{k-n}{2}] \to \mathcal{E}^\Phi[\frac{-k-n}{2}] \quad \text{given by} \quad P_k^\sigma := \Big(- \frac{1}{I^2} I{\cdot}D \Big)^k$$

is a tangential differential operator, and so determines a canonical differential operator $P_k^\sigma : \mathcal{E}^\Phi[\frac{k-n}{2}]|_\Sigma \to \mathcal{E}^\Phi[\frac{-k-n}{2}]|_\Sigma$. For k even this takes the form

$$P_k = \overline{\Delta}^k + \text{lower order terms}.$$

Because $(M, \mathbf{c}, \Sigma)$ determines $\bar{\sigma}$ modulo $+ O(\sigma^{n+2})$, we have:

Theorem

For $k \leq n = d - 1$ the operators P_k are determined canonically by the data $(M, \mathbf{c}, \Sigma)$.

Higher Willmore energies

For suitable regularisations $\overline{M}_\epsilon$ of conformally compact manifolds $\overline{M}$:

$$\mathrm{Vol}_\epsilon = \int_{\overline{M}_\epsilon} \sqrt{g_+} = \frac{v_n}{\epsilon^n} + \cdots + \frac{v_1}{\epsilon} + \mathcal{A} \log \epsilon + V_{ren} + O(\epsilon).$$

Theorem (Graham 2016: arXiv:1606.00069)

If $g_+ = \bar{\sigma}^{-2}\mathbf{g}$, i.e. it is the approximate solution of the sing. Yamabe problem then $\mathcal{A}$ a conformal invariant of $\Sigma \hookrightarrow M$ and

$$\frac{\delta \mathcal{A}}{\delta \Sigma} = \frac{d(d-2)}{2} \mathcal{B}_n$$

So the anomaly term in the renormalised volume expansion provides an **energy** with **functional gradient the obstruction density**, in other words an energy generalising the Willmore energy.

Extrinsic Q-curvature and the anomaly

In fact – also in analogy with the treatment of Poincaré-Einstein manifolds – there is nice local quantity giving the anomaly:

Theorem (G.- Waldron arXiv:1603.07367)

$$\mathcal{A} = \frac{1}{(d-1)!(d-2)!} \int_\Sigma Q$$

where, with $\tau \in \Gamma\mathcal{E}_+[1]$ *a scale giving the boundary metric,* $Q := (-I \cdot D)^n \log \tau$.

- Q here is an **extrinically coupled** Q**-curvature** meaning e.g.

$$Q^{\widehat{g}_\Sigma} = e^{-nf}(Q^g + P_n f) \quad \text{where} \quad \widehat{g}_\Sigma = e^{2f} g_\Sigma$$

and for n even

$$P_n = \Delta_\Sigma^{\frac{n}{2}} + \text{lower order terms}; \;\; P_n \text{ FSA, and } P_n 1 = 0,$$

is an **extrinically coupled** GJMS type operator. Q and P_n are from G.-, Waldron arXiv:1104.2991 = Indiana U.M.J. 2014.

Idea of proof

Use a Heaviside function θ to "cut off" an integral over all $\overline{M}$

$$\mathrm{Vol}_\epsilon = \int_{\overline{M}} \frac{dV^{g_\tau}}{\sigma^d}\theta(\frac{\sigma}{\tau} - \epsilon).$$

Then the divergent terms and anomaly are given by

$$v_k \sim \frac{d^{d-1-k}}{d\epsilon^{d-1-k}} \left(\epsilon^d \frac{d}{d\epsilon} \mathrm{Vol}_\epsilon \right) \Big|_{\epsilon=0},$$

So

$$v_k \sim \int_{\overline{M}} \frac{\delta^{d-1-k}(\sigma)}{\tau^k} \quad \text{and} \quad \mathcal{A} \sim \int_{\overline{M}} \delta^{d-2}(\sigma) I \cdot D \log \tau$$

Then via identities, and the $sl(2)$ again

$$v_k \sim \int_\Sigma (I \cdot D)^{d-k-1} \frac{1}{\tau^k} \quad \text{and} \quad \mathcal{A} \sim \int_\Sigma (I \cdot D)^{d-1} \log \tau$$

An “Obata type” question/conjecture

The Obata Theorem states (more than):

Theorem (Obata)

If g is a metric on the round sphere S^n that is conformal to the standard metric $\bar{g}$ and has constant scalar curvature, then g is Einstein.

NB: The metric here takes the form $g = \sigma^{-2}\bar{g}$ where σ is a smooth **nowhere vanishing function**, and the condition that it has constant scalar curvature is that the scale tractor satisfies $I_\sigma^2 = constant$ (where recall $I = \bar{D}\sigma$). Thus related to the Obata Theorem there is a very nice more general question:

Question

Let $(S^n, \bar{g})$ be the usual round sphere. Let $\sigma \in C^\infty(S^n)$, possibly with $\mathcal{Z}(\sigma)$ non-empty, such that $I_\sigma^2 = constant$. Then is $g = \sigma^{-2}\bar{g}$ necessarily Einstein on $S^n \setminus \mathcal{Z}(\sigma)$?

The Willmore link

Above question is especially interesting if $I^2 = const. \neq 0$ then:

Theorem (G. 2006)

$\mathcal{Z}(\sigma)$, if $\neq \emptyset$, is a smoothly embedded separating hypersurface that gives a conformal compactification of $g = \sigma^{-2}\bar{g}$.

Thus with the earlier mentioned Theorem of G.+Waldron:

Corollary

Let $(S^n, \bar{g})$ be the usual round sphere. Let $\sigma \in C^\infty(S^n)$, with $\mathcal{Z}(\sigma)$ non-empty, such that $I_\sigma^2 = const. \neq 0$. Then $\mathcal{Z}(\sigma)$ is a (higher) Willmore hypersurface.

$\exists$ solutions to $I_\sigma^2 = const. \neq 0$ with $\mathcal{Z}(\sigma)$ totally umbilic. Hence:

Question

Let $(S^n, \bar{g})$ be the usual round sphere. Let $\sigma \in C^\infty(S^n)$, possibly with $\mathcal{Z}(\sigma)$ non-empty, such that $I_\sigma^2 = constant$. Then is $\mathcal{Z}(\sigma)$ necessarily totally umbilic?